Nonlinear meter for the gravitational wave antenna

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Abstract

The principle of a new readout meter for a large-scale gravitational wave antenna is proposed. This principle is based on the registration of the spatial shift of a standing optical wave caused by the gravitation wave inside the Fabry–Perot resonator without absorption of optical quanta.

1. Introduction

Several laboratories at present are involved in the creation of so-called laser gravitational wave antennae (the projects LIGO, VIRGO, GEO-600). The goal of these projects is to detect small perturbations of the metric $h$, which are produced by astrophysical catastrophes (the merging of binary neutron stars, the merging of neutron stars with black holes, etc.) many megaparsecs away from our planet. The first step in these projects is to reach a sensitivity at the level $h \approx 1 \times 10^{-21} - 1 \times 10^{-22}$ (see e.g. Refs. [1–3]). This means that the antennae have to register the small displacements $\Delta L$ between two masses (the mirrors in the optical interferometer) separated at the distance $L$ with the resolution $\Delta L \approx hL/2 \approx 2 \times 10^{-16} - 2 \times 10^{-17}$ cm if $L \approx 4 \times 10^5$ cm. In the following steps of these projects the sensitivity must be substantially higher and it will probably reach the level $h \approx 10^{-23}$ or even a better one. At this level of sensitivity it will probably be possible to detect such features of the gravitational wave burst which will allow one to find the equation of state of neutron stars and even to test general relativity in the ultrarelativistic case.

"Only" two tasks have to be fulfilled in these projects: (i) a sufficient isolation of the test masses from all sources of noise, and (ii) a sufficient sensitivity of the readout meter. There are reasonably optimistic prospects for the isolation of the test masses from the heat bath [4,5]. On the other hand, at present there are no simple realistic schemes for the readout meter which will permit one to reach a sensitivity substantially better than $h \approx 10^{-22}$. There are two main obstacles if the Fabry–Perot resonator with laser pump is used as a meter. The first one (above the standard quantum limit of sensitivity) is of technical origin. The response of the meter in the output phase shift $\Delta \phi_{\text{out}}$ has to be larger than the phase uncertainty of the phase pump,

$$\Delta \phi_{\text{out}} \approx \frac{1}{2} h \omega_{\text{opt}} \tau > \frac{1}{\sqrt{N}},$$

where $\omega_{\text{opt}}$ is the optical frequency, $\tau$ is the averaging time, $N$ is the number of "used" photons. If $h \approx 10^{-22}$, $\omega_{\text{opt}} \approx 2 \times 10^{15}$ s$^{-1}$, $\tau \approx 1 \times 10^{-3}$ s, then $\Delta \phi_{\text{out}} \approx 1 \times 10^{-10}$, $N \approx 1 \times 10^{20}$ photons and the power of the laser
$W_{\text{opt}} \approx \frac{\hbar \omega_{\text{opt}} N}{\tau} \approx 2 \times 10^{11} \text{ erg/s.}$

A small leak of the laser power (for example into the heating of the mirror) may be a serious obstacle [6] in this scheme.

The second obstacle is the SQL: the increase of $N$ in the meter with continuous monitoring of the coordinate will inevitably encounter the standard quantum limit of sensitivity,

$$h_{\text{SQL}} = \frac{2}{L} \sqrt{\frac{\hbar \tau}{m}} \approx 5 \times 10^{-23} \left( \frac{4 \times 10^5 \text{ cm}}{L} \right) \times \left( \frac{1 \times 10^{-3} \text{ s}}{\tau} \right)^{-1/2} \left( \frac{1 \times 10^4 \text{ e}}{m} \right)^{1/2}.$$  

(2)

In this numerical example the necessary power of the laser (in the coherent state!) has to be $\approx 4 \times 10^{12}$ erg/s.

Several schemes of the meter which in principle permit one to overcome the SQL were proposed recently [7–11]. Unfortunately in these schemes the power has to be even larger than in the above example or the possibility of technical implementation of the scheme is not elaborated.

The goal of this article is to describe the principle of a new readout meter for gravitational wave antennae.

2. The readout meter based on nonlinear optical element

The first feature of this meter is the use of a long relaxation time $\tau_{\text{opt}}$ of the optical resonator, which may be much longer than the averaging time $\tau$. In the 40-meter prototype of LIGO at present $\tau_{\text{opt}} \approx 10^{-3}$ s. With the best finesse existing today [12] in the full scale LIGO antenna the value of $\tau_{\text{opt}}$ may be $\approx 10$ s and thus the ratio $\tau/\tau_{\text{opt}}$ may be as small as $10^{-4}$.

The second feature of the meter is the idea not to measure the phase shift of the output beam outside the interferometer but to measure the spatial shift of the standing optical wave induced by the gravitational wave inside the resonator without absorption of photons.

Let us suppose that the Fabry–Perot resonator is created between mirrors A, B, and C (see Fig. 1). The gravitational wave changes the distance between the mirrors and produces a spatial shift of the standing optical wave, which has to be exited by an external source (e.g. by a laser). The amplitude of this shift in the area near mirror B is equal to

$$\delta x = \frac{\delta L_{\text{AB}} - \delta L_{\text{BC}}}{2},$$  

(3)

where $\delta L_{\text{AB}}$ and $\delta L_{\text{AB}}$ are the variations of the distances between the mirrors A, B and B, C, respectively. In the case of an optimal direction and polarization of the gravitational wave the value of $\delta x$ is equal to

$$\delta x = \frac{hL}{2},$$

where $L$ is the unperturbed value of the distances.

The spatial shift of the wave may be measured by a special device mounted on mirror B. This device has to consist of two optical lenses separated by a double focal length (see Fig. 1) and a thin dielectrical plate with cubic nonlinear dielectrical susceptibility $\chi^{(3)}$. This plate has to be situated inside a capacitor part of a microwave resonator. If the plate is located on the slope of the standing optical wave then the shift of the standing wave near B will produce a change of dielectrical susceptibility in the focal zone and therefore a change of capacity and frequency of the microwave resonator,

$$\delta \omega_c = K \frac{\hbar \omega_{\text{opt}}}{2},$$  

(4)
where

$$K = \frac{\pi \chi^{(3)} E_{opt}^2 \omega_c L \sin(\omega_{opt} \sqrt{\epsilon/c})}{\epsilon c \omega_{opt} \sqrt{\epsilon/c}},$$

$E_{opt}$ is the amplitude of the electric field of the optical wave, $c$ is the speed of light, $\epsilon$ is the linear term of the dielectrical susceptibility and $l$ is the thickness of the plate with nonzero $\chi^{(3)}$.

This change of frequency may be registered by the measurement of the phase shift in the microwave resonator (see Fig. 2),

$$\delta \phi_e = \delta \omega_c \tau = K \frac{\hbar \omega_{opt} \tau}{2}.$$  \hspace{1cm} (5)

The device described above evidently has to be a monolithic dielectrical structure with the nonlinear plate created by doping in the way it is done in superlattices.

It is necessary to note that the fluctuations of the optical energy in the resonator will randomly change the value of $\epsilon$ and therefore produce an additional uncertainty of $\omega_c$. This effect limits the sensitivity at the level

$$\hbar \leq \frac{c}{\omega_{opt} L} \sqrt{\frac{\tau}{N_{opt} \tau^*_{opt}}}.$$  \hspace{1cm} (6)

However, it can be avoided by using two dielectrical plates with opposite signs of $\chi^{(3)}$ located symmetrically on the left and right slopes of the standing optical wave.

Comparing formulas (5) and (1) one has to conclude that the dimensionless parameter $K$ defines the efficiency of this meter. Substituting the value $\chi^{(3)} \simeq 1 \times 10^{-14}$ CGSE (fused silica), $E_{opt}^2 \simeq 2 \times 10^7$ CGSE (the optical breakdown of fused silica), $\omega_c \simeq 1 \times 10^{11}$ s$^{-1}$, $l \simeq 1$, $L \simeq 4 \times 10^5$ cm we obtain $K \simeq 1$.

Thus the task is reduced to the measurement of the phase shift in the microwave resonator. The value of the phase shift is approximately the same as the one in the traditional optical scheme. To realize this it will be necessary to use the same number of quanta $N_e \simeq 10^{20}$ in the coherent state (for $\hbar \simeq 10^{-22}$), but the power $W_e$ will be $\omega_{opt}/\omega_c$ times smaller,

$$W_e = \frac{\hbar \omega_c N_e}{\tau} \simeq 1 \times 10^7 \text{ erg/s}. \hspace{1cm} (6)$$

To realize a capacitor in the microwave band for a resonator of the klystron type with $\omega_c \simeq 10^{11}$ s$^{-1}$ it is necessary to have its value in the range of $0.05$-$0.1$ cm. This means that the cross section of the light in the focal area has to be $S_{opt} \simeq 1 \times 10^{-4}$ cm$^2$. To create such a distribution of the light beam it is possible to use a combination of two spherical and two cylindrical lenses. The total optical energy $E_{opt}$ in this case has to be

$$E_{opt} = \frac{L S_{opt} \sqrt{\epsilon E_{opt}^2}}{16 \pi} \simeq 1 \times 10^6 \text{ erg}$$

and the pumping optical power when $\tau^*_{opt} \simeq 10$ s will be

$$W_{opt} = \frac{E_{opt}}{\tau^*_{opt}} \simeq 1 \times 10^5 \text{ erg/s}.$$  \hspace{1cm} (7)

Thus the nonlinear meter described above permits one to obtain a substantial reduction of the necessary optical power with a relatively modest condition for the microwave power.

3. The uncertainty relation in the nonlinear meter

The meter described above is by essence a coordinate one: the phase shift in a microwave cavity is proportional to the displacement of the test masses and thus is proportional to the variation of the metric $h$. If the averaging time $\tau_{meas}$ is longer than the relaxation time $\tau^*_{opt}$ in the cavity, then the resolution in the displacement (3) is equal to
\[ \Delta x_{\text{meas}} = \frac{L}{2\omega_{\text{opt}} K \sqrt{N_c \tau^*_e}}. \]  

(7)

In accordance with the uncertainty relation this measurement has to be accompanied with the perturbation of the canonically conjugated momentum. In the traditional scheme this perturbation is produced by the shot noise of the optical photons which randomly enter and leave the Fabry-Perot resonator. In the discussed meter this effect may be very small (proportional to the ratio \( \tau_{\text{meas}}/\tau_e^* \)), because to get the signal it is not necessary to extract the optical photons from the Fabry-Perot resonator: the measurement is performed inside it without absorption.

A simple calculation (see appendix A) shows that the presence of a dielectrical nonhomogeneity in the optical resonator (a plate located on the slope of a standing wave) produces a redistribution of the e.m. energy in the two parts of the resonator which are separated by this plate. The ratio of the energy densities in these parts and thus the ratio of the ponderomotive forces produced by optical photons which act on the mirrors A and C and on the plate is equal to

\[ \frac{F_A}{F_C} = \frac{n - 1}{n} \sin \frac{\omega_{\text{opt}} \sqrt{\epsilon}}{c}, \]

where \( n \) is the relative value of the nonhomogeneous refraction index.

In the meter discussed above \( n \) is created by the cubic dielectrical nonlinearity \( \chi^{(3)} \),

\[ n = \sqrt{1 + \frac{4\pi \chi^{(3)} E_e^2}{\epsilon}}. \]

Hence

\[ \frac{F_A}{F_C} \sim \frac{4\pi \chi^{(3)} E_e^2}{\epsilon} \frac{\omega_{\text{opt}} \sqrt{\epsilon}}{c}, \]

where \( E_e \) is the strength of the electric field in the capacitor part of the microwave resonator.

The quantum shot noise fluctuations of the energy of the microwave resonator will create a random redistribution of the optical photons in the two parts of the Fabry-Perot resonator (the total number will remain constant). The corresponding fluctuating difference of the forces which act on the mirrors A and C will be equal to

\[ \Delta F = K \frac{\omega_{\text{opt}} \Delta E_e}{\omega_e} \frac{\Delta x_{\text{meas}}}{L}, \]

where \( \Delta E_e \) is the uncertainty of the microwave energy. Due to this force, during the time interval \( \tau \) the momentum which is canonically conjugated to the measured coordinate will be perturbed. The value of this perturbation will be equal to

\[ \Delta p_{\text{pen}} = \frac{\hbar \omega_{\text{opt}} K \sqrt{N_c \tau^*_e}}{L}. \]

(8)

As it follows from formulas (7) and (8) the values of \( \Delta x_{\text{meas}} \) and \( \Delta p_{\text{pen}} \) exactly satisfy the uncertainty relation.

4. On the possibility to overcome the standard quantum limit

As it was mentioned in the introduction to this article there are several schemes of meters which permit one to realize a sensitivity better than the standard quantum limit (SQL) in the gravitational wave antennae on free masses. The key obstacles in the implementation of these schemes are either the necessity to use substantially nonclassical states of the e.m. field or the enormous pumping power or both of them. In the principle of the nonlinear meter described above practically any of the published schemes may be used to obtain a sensitivity better than SQL. The evident advantage is that the implementation has to be done in the microwave band and therefore the power must be much smaller.

Let us consider one concrete procedure of the measurement. In Ref. [9] it was shown that using a “stroboscopic” sequence of coordinate measurements of a free mass it is possible to obtain the resolution in the measurement of an external classical force better than the SQL. The necessary condition for this case is anticorrelation between the additive noise of the coordinate meter and its back action noise. To realize this anticorrelation in the nonlinear meter it is necessary to pump the coherent microwave power not continuously but by shot pulses with a duration \( \tau_{\text{pulse}} \) fulfilling the inequalities

\[ \tau_e^* < \tau_{\text{pulse}} < \tau_F, \]

where \( \tau_F \) is the characteristic duration of the force which has to be detected. The pulses have to be separated by time intervals \( \tau > \tau_F \). The anticorrelation may be obtained by the method proposed in Ref. [10]: by
the tuning of the phase of the reference wave in such a
to that the microwave homodyne gives information
about the sum of the amplitude and the phase quadra-
ture amplitudes in a properly weighted ratio. The min-
imal amplitude of the perturbation of the metric which
may be detected by the use of this method is equal to

$$h_{\text{min}} \simeq \frac{2}{L} \sqrt{\frac{h\tau_{F}}{2m}} \sqrt{\frac{\tau_{F}}{\tau_{\text{meas}}}} = h_{\text{SQL}} \sqrt{\frac{\tau_{F}}{\tau_{\text{meas}}}},$$

(9)

where $\tau_{\text{meas}}$ is the time of extraction of the signal from
the noises and $m$ is the mass of the mirror (see Ap-
pendix B). Hence the "price" which has to be "paid"
for a better sensitivity is the loss of the time resolu-
tion. Another price is the rise of the pumping power:
expression (9) can be rewritten in the form

$$h_{\text{min}} \simeq h_{\text{SQL}} \sqrt{\frac{W_{\text{SQL}}}{W_{e}}},$$

(10)

where $W_{\text{SQL}}$ is the power sufficient to reach the SQL
value and $W_{e}$ is the power used.

The relatively moderate numerical estimate for the
microwave power (see Section 2) shows in our view
that it will be not too difficult to realize a value of
$W_{e}/W_{\text{SQL}}$ smaller than unity and thus to obtain a sen-
sitivity better than $h_{\text{SQL}}$.

Concluding this description of a new principle of a
readout system for the gravitational antenna we want
emphasis that all estimates presented above were
based on a conservative approach: the use of $\chi^{(3)}$ of
a very weak nonlinearity of the fused silica. The use of
a much more nonlinear dielectric will inevitably make
the parameter $K$ larger and thus the meter simpler to
realize.

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Appendix A. Back-action mechanism in the
nonlinear meter

Let us consider a Fabry–Perot resonator with in-
serted dielectrical plate with dielectrical susceptibil-
ity $n^{2}$. Let $\tau_{1}c$ be the optical distance between the
left mirror and the left edge of the plate, $\tau_{2}c/n$ the
thickness of the plate, and $\tau_{3}c$ the optical distance be-
tween the right edge of the plate and the right mirror.
Let $E_{1}^{\text{right}}, E_{2}^{\text{right}}, E_{3}^{\text{right}}$ be the amplitudes of the optical
waves travelling in the right direction in these three
areas, and $E_{1}^{\text{left}}, E_{2}^{\text{left}}, E_{3}^{\text{left}}$ the amplitudes of the waves
travelling in the left direction. These amplitudes satis-
ify the following boundary conditions,

$$E_{1}^{\text{right}} = E_{1}^{\text{left}},$$

$$E_{1}^{\text{right}} e^{i\omega t_{1}} + E_{1}^{\text{left}} e^{-i\omega t_{1}} = E_{2}^{\text{right}} = E_{2}^{\text{left}},$$

$$E_{2}^{\text{right}} e^{i\omega t_{2}} + E_{2}^{\text{left}} e^{-i\omega t_{2}} = E_{3}^{\text{right}} = E_{3}^{\text{left}},$$

$$n(E_{2}^{\text{right}} e^{i\omega t_{2}} - E_{2}^{\text{left}} e^{-i\omega t_{2}}) = E_{3}^{\text{right}} - E_{3}^{\text{left}},$$

$$E_{3}^{\text{right}} e^{i\omega t_{3}} + E_{3}^{\text{left}} e^{-i\omega t_{3}},$$

(A.1)

where $\omega$ is the frequency of the light. From these
equations follows an equation for eigen frequencies of
the resonator,

$$\sin \omega \tau + (n - 1) \sin \omega \tau_{2} \left( \cos \omega \tau_{1} \cos \omega \tau_{3} \right)
+ \frac{1}{n} \cos \omega \tau_{1} \cos \omega \tau_{3} = 0,$$

(A.2)

where $\tau = \tau_{1} + \tau_{2} + \tau_{3}$.

Let us assume that

$$| (n - 1) \sin \omega \tau_{2} | \ll 1.$$

In this case the eigenfrequencies can be presented is
the form $\omega_{k} = \omega_{k}^{(0)} + \nu_{k}$, where

$$\omega_{k}^{(0)} = \frac{\pi k}{\tau}$$

is the zero-order-approximation and

$$| \nu_{k} | \ll \omega_{k}^{(0)}.$$

Substituting this presentation into Eq. (A.2) and omit-
ting the terms of second order one can obtain

$$\nu_{k} \approx - \frac{(n - 1) \sin \omega_{k}^{(0)} \tau_{2}}{\tau} \left( \cos \omega_{k}^{(0)} \tau_{1} \cos \omega_{k}^{(0)} \tau_{3} \right)
+ \frac{1}{n} \cos \omega_{k}^{(0)} \tau_{1} \cos \omega_{k}^{(0)} \tau_{3}.$$
Substitution of this solution into Eq. (A.1) gives the ratio
\[ \frac{E_{\text{right}} e^{i\omega_\text{right} \tau}}{E_{\text{left}} e^{i\omega_\text{left} \tau}} = (-1)^k \frac{1}{2} \left( n - \frac{1}{n} \right) \times \sin \frac{\omega_1^{(0)} \tau_2 \sin \omega_1^{(0)} (\tau_1 - \tau_3)}{2} \]

Hence the ratio of the pressure forces on the right and left mirrors is equal to
\[ \frac{F_{\text{right}}}{F_{\text{left}}} = \left| \frac{E_{\text{right}} e^{i\omega_{\text{right}} \tau}}{E_{\text{left}} e^{i\omega_{\text{left}} \tau}} \right|^2 = 1 + (-1)^k \left( n - \frac{1}{n} \right) \times \sin \frac{\omega_1^{(0)} \tau_2 \sin \omega_1^{(0)} (\tau_1 - \tau_3)}{2} \]

The difference of this forces is equal to
\[ F_{\text{right}} - F_{\text{left}} \approx \hat{F} \left( \frac{F_{\text{right}}}{F_{\text{left}}} - 1 \right) = \hat{F} \left( n - \frac{1}{n} \right) \sin \frac{\omega_1^{(0)} \tau_2 \sin \omega_1^{(0)} (\tau_1 - \tau_3)}{2} \]

where \( \hat{F} = (F_{\text{right}} + F_{\text{left}}) / 2 \).

Appendix B. Stroboscopic measurement of the coordinate of the free mass

Let a classical force \( F(t) \) act on a free mass \( m \). To detect this force the coordinate \( x \) of the mass is measured periodically. The result of such a measurement sequence has the form of a vector,
\[ \bar{x}_j = x_j + \hat{x} + \frac{\hat{p} \tau}{m} + \hat{p}_{\text{fluct}} + \sum_{k=-\infty}^{j} \frac{\hat{p}_{\text{fluct}} (j-k) \tau}{m} \]

where \( \tau \) is the time interval between the measurements,
\[ x_j^{\text{signal}} = \frac{1}{m} \int_{-\infty}^{\tau} F(t') (t - t') \, dt' \]

is the signal, \( \hat{x} \) and \( \hat{p} \) are the initial coordinate and momentum of the mass, \( \hat{p}_{\text{fluct}} \) describe the error of the \( j \)th measurement and perturbation of the momentum during the measurement. We shall assume that the measurements are independent,
\[ \langle \hat{p}_{\text{fluct}} \hat{p}_{\text{fluct}} \rangle = \Delta_p^2 \delta_{jk} \]
\[ \langle \hat{p}_{\text{fluct}} \hat{p}_{\text{fluct}} \rangle = \Delta_p^2 \delta_{jk} \]
\[ \langle \hat{p}_{\text{fluct}} \hat{p}_{\text{fluct}} \rangle = \Delta_{xp} \delta_{jk} \]

and
\[ \Delta_x^2 \Delta_p^2 - \Delta_{xp}^2 = \frac{1}{4} \hbar^2 \]

where \( \Delta_x \) is the error of the measurement, \( \Delta_p \) is the perturbation of the momentum, \( \circ \) denotes symmetrical multiplication.

A simple linear transformation of the output of the meter allows one to exclude the initial values \( \hat{x} \) and \( \hat{p} \),
\[ \bar{p}_j = \frac{m}{\tau} (\bar{x}_{j+1} - 2 \bar{x}_j + \bar{x}_{j-1}) = p_j^{\text{signal}} + p_j^{\text{fluct}} \]
where
\[ p_j^{\text{signal}} = \frac{m}{\tau} (x_j^{\text{signal}} - 2 \bar{x}_j + x_{j-1}^{\text{signal}}) \]
and
\[ p_j^{\text{fluct}} = \frac{m}{\tau} (x_j^{\text{fluct}} - 2 \bar{x}_j^{\text{fluct}} + p_{j-1}^{\text{fluct}}) + p_j^{\text{fluct}}. \]

The signal-to-noise ratio for such a procedure is equal to
\[ \frac{s}{n} = \sum_{j=-\infty}^{\infty} \nu_j p_j^{\text{signal}} \]

where the filtering vector \( \nu_j \) is defined from the matrix equation
\[ \sum_{k=-\infty}^{\infty} B_{jk} \nu_k = p_j^{\text{signal}} \]
and
\[ B_{jk} \equiv \langle \hat{p}_{\text{fluct}} \hat{p}_{\text{fluct}} \rangle \] is the correlation matrix of the total noise,
\[ B_{jk} = \frac{m^2}{\tau^2} \left[ \delta_{j+1k-1} + \delta_{j-k+1} - 4 \left( \delta_{j+1k} + \delta_{j-k+1} \right) \right] \]
\[ + \frac{2m}{\tau} (\delta_{j+1k} + \delta_{j-k+1} - 2 \delta_{j,k}) \Delta_{xx} \]
\[ + \Delta_{xx} \Delta_{pp} \]

Let us use a spectral representation: for any vector \( A_j \)
\[ A(\nu) = \sum_{-\infty}^{\infty} A_j e^{-i j \nu}, \quad A_j = \int_{-\pi}^{\pi} A(\nu) e^{i j \nu} \, d\nu / 2\pi \].
where $\nu$ is the dimensionless “frequency”. The spectral transformation of correlation matrix of the noise has the form

$$S(\nu) = \sum_{-\infty}^{\infty} B_{jk} e^{-i(j-k)\nu},$$

where $S(\nu)$ is the “spectral density” of the noise.

Formulas (B.3)-(B.6) in the spectral representation have the form

$$p^{\text{signal}}(\nu) = \frac{2m}{\tau} (\cos \nu - 1) x_{\text{signal}}(\nu), \quad (B.7)$$

$$\frac{s}{n} = \int_{-\pi}^{\pi} p^{\text{signal}}(\nu) \frac{d\nu}{2\pi}, \quad (B.8)$$

$$S(\nu) \nu'(\nu) = p^{\text{signal}}(\nu), \quad (B.9)$$

$$S(\nu) = \frac{4m^2}{\tau^2} (\cos \nu - 1)^2 \Delta^2 + \frac{4m}{\tau} (\cos \nu - 1) \Delta_{xp} + \Delta_p^2. \quad (B.10)$$

Hence

$$\frac{s}{n} = \int_{-\pi}^{\pi} |p^{\text{signal}}(\nu)|^2 \frac{d\nu}{2\pi}$$

$$= \int_{-\pi}^{\pi} |x_{\text{signal}}(\nu)|^2 (\cos \nu - 1)^2$$

$$\times \left[ \left( \cos \nu - 1 + \frac{\tau \Delta_{xp}}{2m \Delta^2} \right)^2 \Delta^2 + \left( \frac{\hbar \tau}{4\Delta m} \right)^2 \right]^{-1} \frac{d\nu}{2\pi}. \quad (B.11)$$

Let

$$\frac{\hbar \tau}{4\Delta m} \ll 1$$

and

$$\Delta_{xp} = \frac{4m}{\tau} \Delta^2.$$

In this case the expression under the integration in formula (B.11) has a sharp narrow maximum near the value $\nu = \pi$, and the integral is equal to

$$\frac{s}{n} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{16|\pi|^2|\pi|^2}{\tau^2} \frac{d\eta}{2\pi}$$

$$= \frac{16|\pi|^2}{\Delta^2 \Delta^2 + (\hbar \tau / 2\Delta m)^2} \frac{2\pi}{2\pi}$$

where $\eta = \nu - \pi$ and $|\eta| \ll 1$.

Under the same condition the filtering function $\nu(\nu)$ is equal to

$$\nu(\eta) = \frac{-4\tau}{m} x_{\text{signal}}(\pi)$$

and the filtering vector has the form

$$u_j = \frac{4\tau}{m} (-1)^{j+1} x_{\text{signal}}(\pi)$$

$$\times \int_{-\infty}^{\infty} \frac{e^{i\eta}}{\Delta^2 \eta^2 + (\hbar \tau / 2\Delta m)^2} \frac{d\eta}{2\pi}$$

$$= \frac{2(-1)^{j+1} J_{\text{signal}}(\pi)}{h} e^{-\Delta \tau / \hbar} \left( \cos \frac{j}{\eta} + \sin \frac{j}{\eta} \right),$$

where

$$J = 2\Delta \sqrt{\frac{m}{\hbar \tau}}.$$

Hence

$$\frac{s}{n} = \frac{4m}{\hbar \tau^2} \frac{|x_{\text{signal}}(\pi)|^2}{m^2} \tau_{\text{meas}},$$

where

$$\tau_{\text{meas}} = 2J\tau$$

is the characteristic duration of the filtering.

Let the force $F(t)$ have the form of a pulse with duration $\tau_F \sim \tau$ and without coordinate memory. In this case

$$|x_{\text{signal}}(\pi)| \approx \frac{F\tau^2}{2m}$$

and

$$\frac{s}{n} = \frac{F^2 \tau^2 \tau_{\text{meas}}}{m\hbar} = \left( \frac{s}{n} \right)_{\text{SQL}} \frac{\tau_{\text{meas}}}{\tau},$$

where

$$\left( \frac{s}{n} \right)_{\text{SQL}} = \frac{F^2 \tau^3}{m\hbar}$$
is the standard quantum limit value of the signal-to-noise ratio.

References