On the ultimate sensitivity in coordinate measurements

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An analysis of the sensitivity of coordinate meters for ultimate precision mechanical experiments is presented. It is shown that the existing sensitivity of coordinate meters \( \Delta L \approx 10^{-16} - 10^{-18} \) cm/\( \sqrt{\text{Hz}} \) can be substantially improved and reach \( \Delta L \approx 10^{-21} - 10^{-23} \) cm/\( \sqrt{\text{Hz}} \) using new types of microwave and optical parametric transducers based on high-Q dielectric resonators.

1. Introduction

A high-sensitivity meter of small mechanical displacements is a key element in a number of ambitious experimental programs including different types of probe mass experiments, development of gravity-wave detectors and mechanical QND measurements. For example, in gravity-wave antennas, experimentalists are confronted with the necessity to measure small displacements \( \Delta L \approx \frac{1}{2} hL \) of free masses separated by the distance \( L \) due to a gravity-wave induced perturbation of the metric \( h \). For the lowest necessary level of sensitivity \( h = 10^{-21} \), and \( L = 4 \times 10^5 \) cm, the value of the displacement should be \( \Delta L \approx 2 \times 10^{-16} \) cm. In accordance with existing astrophysical predictions for the intensity of gravity-wave signals from extra-terrestrial sources, this displacement should be registered over the time \( \tau \approx 10^{-2} - 10^{-3} \) s, and this requires that the sensitivity of the coordinate meter should be better than \( \Delta L \approx 6 \times 10^{-18} \) cm/\( \sqrt{\text{Hz}} \). A higher sensitivity is necessary for experiments aimed at reaching the standard quantum limit in the measurement of the coordinate of the probe mass \( m \):

\[
\Delta L_{\text{SQL}} \approx \sqrt{\frac{\hbar \tau}{2m}}
\]

\[
= 7 \times 10^{-18} \text{ cm} \left( \frac{10^{-3} \text{s}}{\tau} \right)^{-1/2} \left( \frac{10^4 \text{g}}{m} \right)^{1/2},
\]

(1)

where \( \hbar \) is Planck's constant, \( \tau \) the averaging time. For the numerical values used in estimate (1), the necessary sensitivity of the displacement sensor has to be \( \Delta L \approx 2 \times 10^{-19} \) cm/\( \sqrt{\text{Hz}} \). An even greater sensitivity is required for coordinate meters in the so-called bar antenna in which \( \Delta L_{\text{SQL}} \approx 3 \times 10^{-19} \) cm.

At present, all existing types of high-sensitivity coordinate meters are parametric transducers in which \( \Delta L \) modulates one or several parameters of the e.m. energy flux. In SQUID-based inductance transducers, the best achieved sensitivity to-date is \( \Delta L \approx 3 \times 10^{-15} \) cm/\( \sqrt{\text{Hz}} \) [1]. In parametric transducers based on Fabry–Perot resonators, \( \Delta L \approx (4-6) \times 10^{-17} \) cm/\( \sqrt{\text{Hz}} \) [2,3]. Nearly the same sensitivity is achieved for microwave transducers based on superconducting cavities [4]. Comparing these figures with the projectives given above, one can see that a considerable improvement is required in the sensitivity of existing transducers, or the invention of new types of parametric sensors with a better efficiency of the “displacement-to-energy flux” transformation.

In this paper, we present an analysis of the physical limitations for the sensitivity and new experimental possibilities of the recently proposed coordinate meters based on high-Q dielectric resonators.

2. Coordinate meters on the basis of high-Q dielectric resonators (microwave band)

The sensitivity of a parametric transducer-type coordinate meter based on a high-Q resonator with a
coordinate-dependent resonance frequency may be described by the simple expression (see for example ref. [5])

\[
\Delta L_{\text{min}} \approx \frac{d}{Q_0} \sqrt{\frac{kT}{W\tau}},
\]

(2)

where \(W\) is the e.m. power used to measure the frequency changes of the resonator with the resonance frequency \(\omega_0\) and quality-factor \(Q_0\); \(k\) is Boltzmann’s constant, \(T\) is the noise temperature of the pumping oscillator; \(\tau\) is the averaging time. The parameter \(d\) is used to characterize the dependence of the resonance frequency on the displacement \(AL\),

\[
d = \left(\frac{1}{\omega_0} \frac{\partial \omega_0}{\partial L}\right)^{-1}.
\]

Formula (2) is valid if the averaging time \(\tau\) is large enough: \(\tau \geq Q_0/\omega_0\). For shorter \(\tau\), in formula (2) \(Q_0\) has to be changed to \(\omega_0\). Another condition for the validity of formula (2) is a sufficiently high frequency stability of the pumping oscillator or an adequate bridge circuit (compensation scheme). It is evident from formula (2) that the key value determining the sensitivity is the parameter \(d/Q_0\) measuring the phase response of the transducer per unit displacement.

In superconducting resonators with localized capacity gap, the best demonstrated value of \(d/Q_0\) was \(6 \times 10^{-9}\) cm [4].

Even smaller values of \(d/Q_0\) can be achieved with the employment of dielectric microwave resonators with whispering-gallery modes. In single-crystal sapphire resonators of this type, the quality-factor is limited only by the very small fundamental lattice absorption,

\[
Q^{-1} \approx 1 \times 10^{-8} \frac{\omega}{2\pi \times 10 \text{ GHz}} \left(\frac{T}{70 \text{ K}}\right)^5
\]

in a wide range of temperatures between 40 and 300 K and reaches \(Q_0 \approx 10^{9}\) at \(T \approx 10 \text{ K}\) in the 10 GHz band [6]. Raevsky et al. [7] have noticed that if such a resonator is made in the form of two thin coaxial disks with a small gap between them, then the resonance frequency of whispering-gallery modes is a function of the gap between the two disks, with the equivalent parameter \(d \approx 0.3\) cm for resonators of he 30 GHz band, and no significant degeneracy of the quality-factor at the room-temperature level, \(Q_0 \approx 3 \times 10^4\), is observed. A simple electrodynamical analysis [5] leads to the following formula for the smallest possible value of \(d\) for a system of two coaxial thin disk resonators,

\[
d = \frac{2a}{\epsilon - 1},
\]

(3)

where \(a\) is the thickness of a single disk, \(\epsilon\) the dielectric permittivity of the disk material. Formula (3) is valid if \(a \geq \lambda/\sqrt{\epsilon}\), and if the gap between two disks is less than \(\lambda/2\pi\sqrt{\epsilon - 1}\) (\(\lambda\) is the wavelength). With \(\lambda \approx 0.8\) cm, \(\epsilon = 9\) (sapphire), \(a = 0.05\) cm we obtain the prediction \(d = 1 \times 10^{-2}\) cm. The recently achieved value \(Q_0 = 1.5 \times 10^9\) [8] for a single disk in this frequency band allows one therefore to project \(d/Q_0 \approx 6 \times 10^{-12}\) cm and thus to reach \(\Delta L_{\tau} \approx 1 \times 10^{-15}\) cm/\(\sqrt{\text{Hz}}\) with \(T = 10 \text{ K}\) and \(W = 10^4\) erg/s, which is 1.5 orders of magnitude better than the sensitivity of the best existing transducers. Therefore, the double sapphire disk transducer promises a substantial improvement in the sensitivity of microwave parametric sensors and can be used in a cryogenic bar antenna.

The above estimate may not be regarded as a fundamental limit for coordinate meters of this type. It was shown in ref. [6] that the level of \(Q\) for sapphire resonators at near-helium temperatures is limited by the effect of residual imperfections and doping. In a pure material, intrinsic losses fall as \(T^5\) with the decrease of temperature, and \(Q_0 = 1 \times 10^{13}\) may be projected for \(\omega_0 \approx 2.5 \times 10^{11} \text{ s}^{-1}\) at \(T = 4 \text{ K}\). The corresponding value of the sensitivity would therefore be further improved: \(\Delta L_{\tau} \approx 1 \times 10^{-23}\) cm/\(\sqrt{\text{Hz}}\).

To conclude this section, let us make one important remark. If a coordinate meter is modified in such a way that it can be used for quantum-nondemolition (QND) measurements (e.g., for measurement of one of the quadrature amplitudes), then the ultimate resolution in this measurement will be as smaller than the standard quantum limit, as the relaxation time in the resonator \(\tau_0 = Q_0/\omega_0\) is larger than the averaging time \(\tau\) [9]. For example, if the meter registers the quadrature amplitude of the mechanical oscillator, then the ultimate resolution of the QND procedure will be

\[
\Delta L_{\text{QND}} = \Delta L_{\text{SQL}} \sqrt{\tau/\tau_0^*}.
\]

(4)
For the projected value $Q_0 \approx 1 \times 10^{13}$, the electrical relaxation time will be $\tau_0 \approx 40 \text{ s}$, and with $\tau = 10^{-3}$ s the resolution of the QND procedure will be about $0.005\Delta L_{\text{SQL}}$.

3. Coordinate meters on the basis of high-$Q$ dielectric resonators (optical frequencies)

For optical parametric transducers based on high-$Q$ optical resonators, the ultimate sensitivity is described by the same expression (2) in which $kT$ should be changed to $h\omega_0$:

$$\Delta L_{\text{min}} \approx \frac{d}{Q_0} \sqrt{\frac{h\omega_0}{W_\tau}},$$

i.e. for the given frequency and pump power, the resolution is determined by the characteristic value $d/Q_0$. At the present moment, the only efficient type of optical parametric transducers is the Fabry–Perot cavity. Since the equivalent gap $d$ of Fabry–Perot cavities is evidently equal to the mirror separation $L$, and the quality-factor is proportional to $L$ and to the finesse $F$: $Q_0 = 2dF/\lambda$, where $\lambda$ is the wavelength, the factor $d/Q_0$ is simply equal to the ratio of half the wavelength to the cavity finesse: $d/Q_0 = \lambda/2F \approx 1.5 \times 10^{-11}$ cm, for the best reported finesse $F \approx 2 \times 10^6$ [10]. In the operating parametric sensors of this type, the sensitivity $\Delta L_\tau \approx (4-6) \times 10^{-13}$ cm/√Hz [2,3] has been reported, with the finesse $F \approx (0.5-1) \times 10^5$.

Let us note that the best achieved $d/Q_0$ for a Fabry–Perot cavity is equal to that of a tunable dielectric cavity with an equivalent gap $d = \lambda$ and quality-factor $Q_0 = 10^6$. Comparing these figures with the “microwave” results of section 2: $d_{\text{min}} = 1 \times 10^{-2}$ cm $\approx 10^{-2}\lambda$ at the frequency $\omega_0 \approx 2.5 \times 10^{11}$ s$^{-1}$, and the projected $Q_0 \approx 10^9$, one can reasonably conclude that a vast reserve (several orders of magnitude) does exist for the sensitivity of optical parametric transducers, if the optical analog of the microwave double dielectric cavities be realized. From the electrodynamical point of view, it is evident that similar small values of $d = \lambda$ may be realized, with a minor correction accounting for smaller refraction at optical frequencies. On the other hand, the possibility to realize a high $Q$-factor of whispering-gallery modes in tunable optical dielectric resonators requires a special analysis of scattering losses, and in preliminary estimates, it is appropriate to rely on the reported experimental figures.

$Q_0 = 10^6$ has been reported for a doped-silica integrated-optic ring resonator, embedded in silica substrate [11], with a thickness of the waveguide channel $D = 5 \times 10^{-4}$ cm, and a refraction index $\sqrt{\varepsilon} \approx 1.5$. The possibility to increase the quality-factor to $Q_0 = 1 \times 10^7$ was mentioned. For a pair of coaxial resonators of this type, the equivalent gap in accordance with formula (3) should be $d = 2D/(\epsilon - 1) \approx 4 \times 10^{-3}$ cm. In combination with $Q_0 = 1 \times 10^7$, we obtain $d/Q_0 \approx 4 \times 10^{-10}$ cm, not better than $\lambda/2F$ of the best Fabry–Perot cavities. But it is appropriate to note here that there are no physical limitations preventing the realization of a thinner channel ring resonator with higher $Q_0$, if the appropriate technological procedure be realized to reduce the scattering losses in resonators of this type.

The demonstrated quality-factor of optical whispering-gallery modes in spherical dielectric resonators of fused silica [12] now exceeds $2.5 \times 10^9$ at $\lambda = 0.63 \text{ \mu m}$ [13]. In ref. [5] it was proposed that a spherical resonator cut into two hemispheres could be a sensitive displacement sensor if operated at even $E_{\text{lin}}$ modes having a maximum of the field at the gap between two hemispheres. The predicted equivalent gap describing the tunability of these modes was as small as $d = 4 \times 10^{-4}$ cm, for hemispheres 20 $\mu$m in diameter at $\lambda = 0.63$ $\mu$m. Presuming that the $Q$-factor would be equal to that of an integral sphere, one could project the excellent $d/Q_0 \approx 2 \times 10^{-13}$ cm. This value is 2 orders of magnitude smaller than the equivalent $d/Q = \lambda/2F$ for the Fabry–Perot interferometer with the best reported mirror finesse $F = 2 \times 10^6$ [10]. Unfortunately, preservation of high $Q$ in a double-hemispherical dielectric resonator meets the same difficulties, associated with the necessity to diminish scattering losses due to the residual surface roughness in the gap area.

Another possibility to realize the double dielectric cavity sensor in the optical range has been recently proposed and tested experimentally [14]. The idea of the so-called twin-ball displacement sensor is based on the position-dependent normal mode splitting in the system of two coupled spherical resonators. By contrast to the double-hemispherical (and double-
disk) design, in which the operational strongly tunable mode is not present in separate resonator "halves", in the twin-ball sensor, the mechanical tuning is provided by a coupling of initially close-frequency modes in two separate integral spheres each possessing a high $Q$-factor.

The authors of ref. [14] have experimentally demonstrated the value $d/Q_0=2\times10^{-9}$ cm, equivalent to that of a Fabry–Perot cavity with good supercoating mirror finesse $F=2\times10^4$. The analysis of the normal mode splitting in two spherical resonators coupled via an evanescent field in their equator area, yields the following estimate for the minimal $d$ characterizing the mechanical tuning [14],

$$d \simeq C(\epsilon) \frac{\lambda}{2\pi \sqrt{\epsilon - 1}} l^{3/2} = 1 \times 10^{-2} \text{ cm},$$

where $l=\pi D\sqrt{\epsilon / \lambda}$ is the mode index; $D=10$ μm is the smallest diameter of dielectric spheres preserving radiative $Q$; $C(\epsilon)$ is the numerical factor equal to 1.4 for the considered type of $E_{11}$ modes; $\lambda=0.63$ μm, $\sqrt{\epsilon}=1.457$ is the refraction index of fused silica. In combination with the achieved $Q_0=2.5\times10^9$ for the single sphere, we obtain $d/Q_0 \approx 4 \times 10^{-12}$ cm, an order of magnitude better than in the best existing FP cavities. The $Q$-factor of spherical whispering-gallery microresonators is fundamentally limited by intrinsic absorption in fused quartz. The minimal value of losses realized in optical fibers allows $Q \approx 1 \times 10^{11}$ at $\lambda=1.55$ μm, with a molecular-size roughness of the resonator surface (formed by the free effect of the surface tension upon fusing silica balls). With this value of $Q_0$, and the minimal $d \approx 2.5 \times 10^{-2}$ cm for a larger wavelength, we obtain that the twin-ball transducer of two 25 μm spherical whispering-gallery microresonators can have $d/Q_0 \approx 1 \times 10^{-12}$ cm.

With 10 mW of pump power, according to formula (5), the twin-ball optical sensor should have a sensitivity $\Delta L \approx 1 \times 10^{-21} \text{ cm/Hz}$, which is about 5 orders of magnitude better than in the best existing optical transducers based on FP cavities.

It should be also mentioned that in contrast to microwave sensors, a high sensitivity of coordinate meters on the basis of optical dielectric cavities can be realized under room temperature conditions.

4. Conclusion

The simple estimates given above, in the authors' opinion, prove the perspective of coordinate meters on the basis of high-$Q$ dielectric resonators, both in the microwave and in optical frequency ranges, and demonstrate the presence of a substantial reserve in sensitivity of parametric displacement transducers. It is necessary to note, however, that the double dielectric cavity sensor can be used only in the case when probe masses are located close to each other, and the separation between the moving parts of the dielectric cavity can be made small. Therefore high-sensitivity transducers of this type may not be employed in full-scale free-mass gravity-wave antennae.

There exists an additional possibility to increase the sensitivity of a parametric-type coordinate meter: to use nonclassical (squeezed) states of the e.m. field for pumping the transducer, or to prepare nonclassical states in the meter resonator itself. This latter possibility may be easily realized in high-$Q$ optical dielectric cavities. Due to the strong localization of the e.m. field, these resonators exhibit nonlinear properties at a low level of the pump power [12]. However, a correct analysis of the possible improvement of the sensitivity when nonclassical quantum states are used in high-$Q$ resonant parametric transducers, requires a special detailed analysis.

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References


