Investigations of the dynamics and mechanical dissipation of a fused silica suspension

Phil Willems a,∗, Virginio Sannibale a, Jaap Weel a,1, Valery Mitrofanov b

a LIGO Project, California Institute of Technology 18-34, Pasadena, CA 91125, USA
b Department of Physics, Moscow State University, Moscow 119899, Russia

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Abstract
The quality factor (Q) of a violin mode of a fiber under tension is related to the Q of the unloaded fiber by the dilution factor. We calculate this dilution factor from measurements of the Q’s and compare to measurements based on the shift of the violin mode frequencies with temperature, and to theoretical predictions. We also report supporting measurements of the temperature dependence of the Young’s modulus of fused silica. The Q’s of the violin modes are the highest yet measured in fused silica mechanical resonators. © 2002 Published by Elsevier Science B.V.

1. Introduction
The analysis of the frequencies of the violin modes of fibers under tension began with Pythagoras 26 centuries ago [1]. His work represents one of the earliest triumphs in the application of mathematical principles to physical phenomena. Research to understand this system persists to this day.

The thermal noise motion of the suspension wires in interferometric gravitational wave detectors sets a fundamental limit to their sensitivity [2]. Thermal noise motion can be shown to increase with mechanical dissipation by the fluctuation–dissipation theorem [3]. For this reason, advanced detectors will use materials with very low mechanical loss, such as fused silica, for these wires [4–6]. The mechanical loss of a fiber suspension can be even lower than the loss of the material it is made of, due to the dilution factor, which may be understood roughly as the ratio of the restoring force due to tension, which is lossless, to that due to the fiber stiffness, which is dissipative [2].

Verification of the very low loss in fused silica suspensions has occupied the efforts of many laboratories over the past decade [7–14], with results generally falling short of what would be expected from the intrinsic loss of the material and the dilution factor. There are many possible causes: it can due to dissipation of the oscillation energy through the suspension point, or recoil damping; it can be due to damping by collisions with surrounding gas atoms. It may also be due to a stress dependence of the mechanical loss of fused silica. Nonlinear thermoelastic damping, or NTE [15], is one such stress-dependent loss mechanism that has recently been identified. It may also be true that our theoretical understanding of the dilution
of mechanical loss by the tension in the suspension wire is not correct.

In this Letter we report on investigations we have made in our laboratory with various fused silica suspensions in an effort to better understand and quantify their dynamics as they relate to mechanical dissipation.

2. The frequencies of the violin modes

The bending of suspension fiber under tension is governed by the dynamic beam equation

\[ EIX^{(iv)}(z, t) - PX''(z, t) = \rho_L \ddot{X}(z, t), \]  
where \( E \) is the Young’s modulus, \( I = \pi r^4/4 \) is the fiber bending moment of inertia, \( P \) is the tension, and \( \rho_L \) is the fiber’s linear mass density. A violin mode of this suspension fiber will exhibit sinusoidal motion, and so we may separate the variables in the usual way to obtain the ordinary differential equation

\[ EIX^{(iv)}(z) - PX''(z) + \rho_L \omega^2 X(z) = 0, \]  
where \( \omega \) is the angular frequency, with solution

\[ X(z) = A \cos(k_z z) + B \sin(k_z z) + C \cosh(k_e z) + D \sinh(k_e z), \]  
where

\[ k_t = \sqrt{-P + \sqrt{P^2 + 4EI\rho_L \omega^2}} / 2EI \]
\[ k_e = \sqrt{P + \sqrt{P^2 + 4EI\rho_L \omega^2}} / 2EI. \]  

If this suspension fiber, having length \( L \), is connected to relatively heavy masses at top and bottom, then to an excellent approximation we may assume they are fixed, and then the boundary conditions on the fiber are \( X(0) = X'(0) = X(L) = X'(L) = 0 \). These four boundary conditions relate the constants \( A, B, C, D \) to one another. A nontrivial solution of the equations exists only if the determinant of the matrix of coefficients of \( A, B, C, D \) equals zero. This characteristic equation determines the eigenfrequencies \( \omega_n \) of the violin modes.

The characteristic equation is the following complicated transcendental equation:

\[ 2kek_t \left[ 1 - \cos(k_t L) \cosh(k_e L) \right] \]
\[ + (k_t^2 - k_e^2) \sin(k_t L) \sinh(k_e L) = 0, \]  
but it is possible to use the facts that the lowest mode frequencies are very close to those of a perfect string and that \( k_e L \sim 100 \gg 1 \) to make the very good approximation

\[ fn = \frac{n}{2L} \sqrt{\frac{P}{\rho_L} \left[ 1 + \frac{2}{k_e L} + \left( 4 + \frac{(n\pi)^2}{2} \right) \frac{1}{(k_e L)^2} \right]} \]

for the frequency \( fn = \omega_n/2\pi \) the \( n \)th violin mode. The first part of \( fn \) is just the frequency of the violin mode of a perfect string, for which all the restoring force is due to the tension. The additional factors involve \( k_e \) and quantify the additional restoring force due to the stiffness \( EI \) of the fiber. There will be a small correction to Eq. (6) due to the recoil of the (nearly) free masses at the ends of the fibers. However, for our suspension this correction can easily be estimated to be only \( \sim 0.001\% \) for the first mode and to fall as \( n^2 \) with mode number, so we can safely ignore it.

The dilution factor \( D_n \) of the \( n \)th violin mode satisfies the equation

\[ \phi_n = \phi_{int}/D_n, \]  
where \( \phi_n \) and \( \phi_{int} \) are the losses of the violin mode and the fiber itself, respectively. The dilution factor for the mechanical loss of the \( n \)th violin mode is shown in Appendix A to be given by

\[ D_n^{-1} = \frac{2}{k_e L} \left[ 1 + \left( 4 + \frac{(n\pi)^2}{2} \right) \frac{1}{(k_e L)^2} \right]. \]  

We see immediately that the anharmonicity of the mode frequencies is closely related to the dilution factor. This is not surprising when we consider that the dissipation in the fiber is almost entirely due to the bending and not due to the work done against tension. We might therefore expect that careful measurement of the frequencies of the modes would provide some confidence that we understand the suspension dynamics, and thereby the dilution factor.

We used a version of the apparatus used by Braginsky et al. [9], in which the suspension fiber is welded.
between upper and lower fused silica pendulum bobs. Our apparatus is shown in Fig. 1. The cylindrical upper bob was itself suspended by two loops of 0.006" steel music wire from a rigid clamping structure. These wires were polished and lubricated to minimize recoil losses. Thus the whole system was a double suspension, and the suspension fiber part of a seamless monolithic fused silica structure. We drew a fiber from a 3 mm diameter rod of Suprasil 2 fused silica in a premixed hydrogen/oxygen flame on an automated glass-working lathe and cut it to a length of approximately 22 cm. The diameter was uniform over the length of the fiber to within a few percent. We welded the fiber directly onto short cylindrical pins welded to nipples machined into the bobs. We then annealed the fiber and nipples after welding with a cooler hydrogen diffusion flame.

We excited the violin modes with an electrostatic drive (ESD) consisting of a high-voltage wire held near enough to the fiber to expose it to a strong electric field gradient. The displacement of the fiber as it vibrated was monitored by using a split photodiode to measure the position of the shadow cast by the fiber in a HeNe laser beam. The frequencies were measured by beating the fiber displacement signal against a known stable reference frequency \( \sim 0.1 \text{ Hz} \) away using a lock-in amplifier. The frequencies of the violin modes at room temperature are plotted in Fig. 2 along with a fit to the mode frequencies. In making the fit we assumed a uniform fiber of length 22.2 cm (the measured value) rigidly clamped at both ends with tension 20 N (due to the lower mass of 2.04 kg), and allowed the fiber radius to vary. The best fit value is \( r = 157 \mu \text{m} \), which compares very well with the 315 ± 4 \( \mu \text{m} \) diameter measured using a micrometer. The measured mode frequencies are all within 0.2% of the fit frequencies, indicating that the uniform fiber assumption and Eq. (6) are quite good. Each mode listed is actually a doublet of two closely spaced modes, corresponding to orthogonal polarizations. For the 780 Hz fundamental, the splitting is 0.1 Hz. This splitting, too small to be seen in the figure, is likely due to a slight ellipticity of the fiber or nonaxisymmetric welding of the fiber to the bobs. The recoil of the masses for this mode is so small that the nonaxisymmetric form of the upper mass cannot be responsible for the splitting.

The reader has no doubt noticed that the predicted frequencies, while very close to the measured values, show a slightly weaker dependence on mode number than do the data, and has postulated that perhaps the agreement could be better if some physical parameters were adjusted. We did in fact find that a slightly shorter and thicker fiber would replicate the data much better, but that this length and thickness were excluded by independent direct measurement. We could also better fit the data with the given fiber length and radius if we adjusted the Young’s modulus from our assumed value of 74.5 GPa to about 100 GPa. This possibility should
be considered, since the compressibility of fused silica is known to increase at high pressure [16] (we in fact have adjusted our estimate of $E$ upwards about 2% relative to that for unloaded fibers to account for this effect). However, we could not find evidence in the literature for so large a dependence.

3. Measurement of the vertical bounce frequency

Moreover, we performed an experiment that measured the Young’s modulus of the strained fiber. Generally, when we excited a violin mode to a high amplitude and viewed it on a spectrum analyzer, there were sidebands visible around the violin mode frequency. These sidebands were symmetrical around the violin mode and, depending on their amplitude, could be seen to third order. These sidebands came and went due to unknown factors (presumably seismic excitation), but their amplitudes relative to the carrier were consistent with phase modulation, and their presence was insensitive to the precise alignment of the probe laser on the fiber, indicating that they were truly present in the fiber motion. A sample spectrum in which the sidebands are especially strong is shown in Fig. 3.

If the lower mass bounces up and down (as it did in our suspension, since there was little isolation of vertical motion from the ground), then the tension in the suspension fiber will be modulated at the vertical bounce frequency: $P(t) = P_0 + \delta P \cos(\Omega t)$, where $\Omega$ is the vertical bounce frequency. We may then plug this time-varying $P$ into Eq. (6) to see that the violin mode frequency is phase-modulated at frequency $\Omega$.

The fiber length $L$, and the parameters related to the length such as $\rho L$, will also be time-varying, but this is a second-order effect compared to the variation in $P$. Thus, the sidebands on the violin modes provide a convenient measure of the vertical bounce frequency [17]. In our case this frequency was 16.6 Hz.

The calculation of the expected vertical bounce frequency is not completely straightforward because the vertical spring formed by the suspension fiber was coupled to the vertical spring formed by the upper mass suspension. Using a software model developed for LIGO called the E2E-MSE code [18], we modelled the full suspension and found that the vertical bounce frequency was as measured if the Young’s modulus had our assumed value of 74.5 GPa.

Such detailed analysis is not truly necessary. In fact, the radius of the fiber was not perfectly uniform, but showed about 1% variation over its length, which could easily explain the discrepancies seen. Also, we have made the assumption that $E$ is independent of frequency. This is also not perfectly true.

The agreement between the measured and predicted resonance frequencies provides reassurance that the dynamics of the suspension are understood. However, as a means of predicting the dilution factor they leave much to be desired. The fraction of the violin mode frequency that originates in the fiber stiffness is about $2/keL$, which for this fiber was only about 1%. The change in this fraction for the different modes was even smaller, about $(\pi/keL)^2/2 \sim 0.1\%$. As mentioned above, small nonuniformities in the fiber diameter will create mass concentrations along the fiber that will shift the various mode frequencies by differing amounts in a way that can mask the variation due to fiber stiffness. What we require is a way to vary the fiber stiffness independently of the other parameters.

There is a way to do this. In Appendix B we show that the temperature dependence of the mode frequencies obeys the following formula:

$$\frac{df_n}{dT} \approx \frac{1}{2}(\alpha - u_0 \beta) + \frac{1}{D_n}[\alpha + u_0 \beta + \beta/2].$$

(9)
where $\alpha$ is the linear thermal expansion coefficient, $\alpha_0 = P/AE$ is the static strain of the fiber with $A$ its cross-sectional area, and $\beta = (dE/dT)/E$ is the temperature dependence of the Young's modulus of fused silica. The first term in Eq. (9) can be understood as the frequency shift due to the change in the overall length of the fiber. The second can be understood as the shift due to the change in the stiffness of the fiber. For fused silica, $\alpha \approx 5 \times 10^{-7}$ K$^{-1}$, which is much smaller than $\beta \approx 2 \times 10^{-4}$ K$^{-1}$. If we set the fiber strain such that $\alpha - \alpha_0 \beta = 0$, then Eq. (9) takes the very simple form

$$\frac{df_n}{dT} \approx \frac{\beta}{2D_n}.$$  

(10)

With knowledge of $\beta$, therefore, we may easily obtain the dilution factors from measurement of the temperature dependence of the mode frequencies.

4. Measurement of $(dE/dT)/E$ of fused silica

There are many measurements of $\beta$ in the published literature [19–25]. However, the results differ from each other by a factor of about 3, and there is even a report that differs from the others in the sign of $\beta$ [24]. In order to have a useful model of the suspension to compare with the data it is necessary to know $\beta$ more precisely than this. We abstain from any critical evaluation of the published values, and simply take them at face value. Since the reason for the variation of the published results may reflect actual variation in the fused silica samples used, or in the frequency of the measurement, or in the manner in which $\beta$ was measured, we felt it safest to measure $\beta$ for the type of fused silica we use, Suprasil 2 fused silica provided by Heraeus, prepared identically to our suspension fiber, for bending motion in the same frequency range.

Prior to building the double pendulum described above, we drew a fiber of diameter 165 $\mu$m in the same manner as the suspension fiber and cut it to a length of approximately 10 cm, leaving one thick rod end attached. We then mounted it by this thick rod end in a rigid clamp with indium gaskets under vacuum and measured its free vibration frequencies using the same HeNe laser shadow sensor and split photodiode as the fiber motion sensor. We excited the modes with the same electrostatic drive.

To vary the temperature of the fiber the whole was surrounded by a radiative heater. This radiative heater consisted of a thick-walled copper tube with one end closed and the other having a cap that allowed the fiber and part of the rod end to protrude inside without touching. A pair of slits ran lengthwise down the tube to provide entry for the sensing laser beam and ESD. A pair of resistive heaters were glued to the sides of the tube, and thermometers inside the tube monitored the temperature. To increase its efficiency, the inside of the tube was lined with black plastic for high emissivity and the outside was polished for low emissivity.

The free bending mode frequencies of a cantilever of length $L$ clamped at one end satisfy the equation

$$\cosh(\kappa L) \cos(\kappa L) + 1 = 0,$$

(11)

where $\kappa = \sqrt{\alpha^2 \rho L/EI}$. This equation is satisfied by some set of constants $\kappa_i L = C_i$, where $i$ labels the mode. This set of equations can be solved for the dependence of the mode frequencies $f_i$ on $\beta$ to get

$$\frac{df_i}{dT} = \frac{1}{2} \frac{dE/dT}{E} \cdot \frac{\alpha}{\beta} = \frac{\beta - \alpha}{2}.$$  

(12)

In deriving Eq. (12) we have considered the variation of $L$ due to $\alpha$ as well as those of $\rho_L$ and $I$ due to $\alpha$ through the radius $r$. Note that the dilution factor does not appear in this formula. This is because the fiber is not under stress. In practice, we found that the mode frequencies were not in good agreement with the prediction for a cantilever, presumably due to the variation in diameter of the fiber near the clamped end, where it widens to 3 mm. This merely changes the values of the $C_i$'s, however, and does not affect the dependence of $f$ on $T$. When the fiber diameter is precisely controlled the frequencies are very close to their predicted values.

Fig. 4 shows the frequencies measured as the oven temperature was varied from 20 to 50$^\circ$C. Although this is a very small temperature range compared to the previously published experiments, we require knowledge of $\beta$ only around room temperature, and the precision of our measurements allowed us to do this. In order to verify that the frequencies were reproducible and not affected by contamination depositing on the fiber, we collected the data in random order and looked for scatter in data taken at similar temperatures after the oven was cycled. No systematic dependence of mode frequency on time within the oven was found.
Another systematic error that we watched for was the frequency shift due to damping in the fiber. A velocity-dependent friction (such as residual gas damping) will shift the resonant frequency by a fraction equal to the magnitude of the damping fraction. If this friction itself varies with temperature it can be mistaken for a frequency shift due to $\beta$. Our data are negligibly affected by any such sources of friction because the loss factor of the modes is much smaller than the fractional error in measured frequency due to finite measurement time, which is 1 mHz.

These data are all consistent with a value for $\beta$ of $1.52 \times 10^{-4} \text{ K}^{-1} \pm 10\%$. The error is dominated by uncertainty in the fiber temperature. A subsequent experiment with another, identically prepared fiber gave the same value within errors. The value of $\alpha$ is only $\sim 5 \times 10^{-7} \text{ K}^{-1}$ and introduces only a 1% correction into Eq. (12). This value of $\beta$ is within the range of values previously published, if somewhat low.

5. Measurement of the violin mode frequencies vs. temperature

Given that we knew $\beta$, we could then proceed to measure $D_n$. We enclosed the entire double pendulum in a larger, cleaner oven consisting of two aluminum half cylinders studded on the outer surface with resistive heaters and on the inside with oxidized copper for high emissivity. The temperature in the vicinity of the fiber was measured by a suspended thermistor. Prior to installing the suspension in the oven the uniformity of the temperature within the oven was measured with additional thermistors to be $\sim 10\%$.

The dependence of the first five mode frequencies on temperature is shown in Fig. 5; the sixth mode proved to be difficult to excite and we could not collect data at enough frequencies to include it. As the data show, the frequency shifts are small compared to the polarization splitting of the modes, so it was necessary to carefully verify which mode of the doublet was being measured. This we did by performing a Fourier transform of the ringdown signal for each frequency measurement and locating both peaks. We were not able to reliably excite both modes of a given doublet independently to a sufficiently high level for accurate frequency measurement, except for the first and fourth modes. However, for these two modes the dependence of frequency on temperature was nearly polarization-independent.

The data for each mode is well fit by a simple linear slope. The value of $D_n$ corresponding to the slope for
each mode is plotted in Fig. 6, along with the prediction of Eq. (8) using the same value of the fiber radius derived from the fit to the frequency data in Fig. 2. In estimating \( D_n \) from the data, we use the full equation (9), rather than the simpler approximation (10); however, this changes the estimate only by \( \sim 7\% \) for the lowest mode and less for the others. The value of \( \alpha \) used is derived from fiber \( Q \) measurements described later in this Letter. The uncertainty in \( D_n \) is about half due to the uncertainty of the temperature of the fiber and half due to the standard error in the fit. The agreement is quite good, better than expected given the errors of the measurement, probably reflecting the fact that the temperature errors are more systematic than statistical.

We cannot resist making a small digression here. It is clear from Eq. (9) that there is a value of the strain \( u_0 \) for which the frequency of a violin mode is independent of temperature. Depending upon the value of the dilution factor for that mode, this strain can be well below the breaking strength of the fiber. Such a suspension, made from very pure fused silica, would be a high-\( Q \) temperature-stable mechanical resonator, and one that is sensitive to the gravitational acceleration \( g \). This might be a useful device for gravimetry.

6. Measurement of the violin mode quality factors

Because it introduced contamination that kept the pressure in our vacuum chamber above \( 10^{-6} \) Torr, we removed the oven from the vacuum when measuring the quality factors of the suspension. Apart from this, the measurement of a quality factor was no different experimentally than the measurement of a mode frequency. The data, a 0.1 Hz sine wave taken from the lock-in output, was processed in the following way. First, the data was averaged over the entire set and this average removed from each data point. This removed a constant output offset of the lock-in from the data. Then the absolute value of each data point was taken. Then the entire data set was smoothed by taking a sliding average of \( \sim 1\% \) of the data points. Finally, a decaying exponential was fit to the resulting curve.

The data thus obtained are shown in Fig. 7. Only the highest data point for each mode is shown, but each point was reproducible within 10%, which we take to be the accuracy of the measurement. The
measured $Q$'s did not vary with the power of the probe HeNe laser beam from 30–300 µW, thus excluding radiometric effects on the fiber. They also did not depend on whether the ESD was energized with a DC voltage or not. The $Q$'s were independent of pressure from $3 \times 10^{-6}$ to $3 \times 10^{-7}$ Torr, except for the first mode, which increased from $3.5 \times 10^8$ to $4.4 \times 10^8$. No dependence on the amplitude of the fiber excitation was observed, and using the spectrum analyzer we saw that before we excited the modes with the ESD, their background excitation was at least 60 dB below the level used in the measurement, excluding seismic motion as a source of error.

Fig. 7 also shows some theoretical predictions for the violin mode $Q$'s. These require measurement of the intrinsic fiber $Q$, which we now describe.

7. Measurement of the unloaded fiber quality factors

Having measured the violin mode $Q$’s, we broke the suspension fiber about 10 cm from the upper mass with pliers and measured the quality factors of its cantilever modes. This time the mode frequencies agreed quite well with a fit of Eq. (11); all modes were within 0.2% of the predicted values using the value of the fiber radius obtained by the violin mode frequency data, with the cantilever length as the only free fitting parameter. The fit length was 9.59 cm. The frequency data are shown in Table 1 along with the predictions of the model.

The quality factors of the unloaded fiber are shown in Fig. 8. They show a clear asymptotic value at high frequency of $Q \approx 1.1 \times 10^7 \pm 10\%$, as well as a reduction at low frequency that is characteristic of thermoelastic damping. Unfortunately, this cantilever had no resonant frequencies below the thermoelastic damping peak, and the first mode, which lies more or less directly atop the peak, was too highly excited by seismic motion in the laboratory to permit measurement of its quality factor. Nevertheless, if we assume typical values for the heat capacity per unit volume $C_v$ and thermal diffusivity $D_{th}$, thus fixing the peak frequency of the damping, then a fit of thermoelastic damping plus a frequency-independent loss $\phi_0$ to the data gives $\alpha \approx 3.9 \times 10^{-7}$ K$^{-1}$ and $\phi_0 = 7.6 \times 10^{-8}$. This value for $\alpha$ is lower than expected based on handbook values. Thermoelastic damping lower than expected has been observed previously by Huang and Saulson [26] in unloaded metal wires. In their case, the discrepancy was due to the twisted shape of their sample, which caused part of their restoring force to come from shear, which is free of linear thermoelastic damping. Our fiber was visibly straight, and the mode frequencies fit the straight cantilever model too well to allow for any significant additional restoring forces besides pure bending. It may be true that the values of
$C_v$ and $D_{th}$ of this fiber differ from the standard values for fused silica, especially in the surface layer. However, the variation of $\alpha$ in different samples of fused silica is well documented.

Knowing the radius of the fiber and the temperature dependence of its Young’s modulus, along with the thermal expansion coefficient, we can calculate the losses for the modes due to the linear and nonlinear thermoelastic effects. The quality factor for a violin mode is given by Eq. (7). We take as the intrinsic loss $\phi_{int}$ of the fiber our fitted high frequency value $\phi_{int} = 7.6 \times 10^{-8}$ plus the loss due to nonlinear thermoelasticity, which is given by the formula [15]

$$\phi_{NTE} = \frac{E(\alpha - u_0 \beta)^2 T}{\rho C_v} \frac{\omega \tau}{1 + (\omega \tau)^2},$$

where $\tau = 4 r^2 / 13.55 D_{th}$; this loss can be calculated using the measured values for $\alpha$ and $\beta$. For the value of $u_0$ in this suspension this loss nearly cancels to zero (the condition for null nonlinear thermoelastic damping is exactly the same as the condition that the first term in Eq. (9) go to zero). The predicted quality factors based on this assumed loss for the first five modes are plotted in Fig. 7, along with the measured $Q$’s at room temperature. In addition, we plot the expected quality factors in the case where $\beta = 0$, to show the amount of difference introduced by NTE. We also plot a fit of LTE to the data where $\alpha$ is optimized as a free parameter. The best fit to the data occurs when $\alpha = 5.9 \times 10^{-7}$ K$^{-1}$. This value, while not consistent with the $Q$’s for the unloaded fiber, is not implausible for fused silica.

The data are generally lower than expected. However, they agree better with the linear thermoelasticity hypothesis than the nonlinear thermoelasticity hypothesis, especially if the larger value for $\alpha$ is adopted. We argue against the interpretation that this data refutes the NTE theory, however. There are three possible reasons why the data could disagree with the NTE theory: the theory could be wrong, the theory could be incomplete, or there may be excess loss from the system. We see no reason why the theory should be wrong. However, it could be that the material parameters of fused silica at high stress are not the same as those as those used in our model. For example, the intrinsic loss of fused silica at high stress could be higher at low frequencies, independent of the thermoelastic damping. Such a hypothesis could be tested by measurement of the losses of the vertical bounce mode, which will not have a thermoelastic component. Such a measurement was not possible with this apparatus due to poor isolation of the vertical bounce mode from losses in the support structure. A stress dependence of $\alpha$ might also be the cause. It is conceivable, though unlikely, that $\alpha$ and $\beta$ are modified by the high stress in the fiber in such a way as to fit with the data.

The most likely explanation is excess losses in this system extrinsic to the fiber, such as recoil losses in the suspension structure or friction in the upper suspension. The violin mode $Q$’s measured in this experiment are several times higher than typically measured values, and more than twice as large as the next highest values. To expect that these values are completely free of excess losses would be to expect unprecedented precision in this experiment, and cannot be assumed prima facie.

The interpretation of our $Q$ measurements must therefore be conditional. If the NTE theory proves to be correct, then our measured $Q$’s fall short of expectations, though for the two highest modes the difference is only about 25%. Given that the NTE prediction itself is uncertain at the $\sim 10\%$ level since it relies on measurements of $\alpha$, $\beta$, and the unloaded fiber $Q$’s, this is fairly good agreement for those modes. Certainly, if we simply use the measured high frequency value of $Q \approx 1.1 \times 10^{-7}$ when generating the NTE predictions, the discrepancy nearly vanishes. At lower frequencies there is still a factor of two difference between theoretically expected $Q$’s and those measured that must be explored. The NTE theory is as yet unconfirmed, however, and the data are consistent with linear thermoelastic damping using a reasonable value

### Table 1

<table>
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<tr>
<th>Mode</th>
<th>Measured frequency (Hz)</th>
<th>Predicted frequency (Hz)</th>
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<tr>
<td>8</td>
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<td>4336.7</td>
</tr>
</tbody>
</table>

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for $\alpha$. Clearly, the verification of nonlinear thermoelastic damping must be a high priority to those working in this field.

8. Discussion and conclusions

This research has shown that, if care is taken to precisely fabricate a fused silica suspension, then much of its dynamical behavior is consistent with the standard dynamic beam equations. Certainly, the mechanical dilution factors, as measured by the shifts in the mode frequencies, agree well with theory. In addition, the violin $Q$‘s in this experiment are the highest mechanical $Q$‘s yet measured in a fused silica suspension (about 2 times higher than the next highest value [11]), and the unloaded fiber $Q$‘s are significantly better for their radius than reported in other experiments; Gretarsson and Harry measured $Q \sim 5 \times 10^6$ for fibers of similar radius [29]. We take a moment to discuss why this might be so.

First, the use of precisely fabricated fused silica suspensions allowed us to analyze the dynamics of our suspension with a precision not before demonstrated. This is an essential improvement. Most other research has used fibers freshly drawn from thicker rods, as did we. However, these other experiments generally left the thicker rod ends on the fiber for ease in welding to the suspension masses. Thus, those fibers tapered gradually to larger diameter at their ends. It has been shown by numerical analysis that these tapers could significantly reduce the $Q$‘s of violin modes [27], and in a way that is hard to quantify because this taper is not a controlled parameter. Without being able to precisely specify the radius of the fiber where it bends, it is simply not possible to analyze the mode frequencies, their dependence on temperature, the expected level of thermoelastic damping, or the expected dilution factor.

Second, we note that our fibers are drawn and welded using clean hydrogen/oxygen and hydrogen torches. In addition, the drawing torch is itself manufactured from fused silica. Some earlier experiments used natural gas torches with metal tips. Our technique will tend to minimize the deposition of hydrocarbons and metallic impurities on the fiber surface [28]. It is very likely that surface effects dominate the mechanical dissipation of thin fused silica fibers [29].

Third, we note that after welding the fiber into the suspension with a premixed hydrogen/oxygen flame, we annealed it with a cooler pure hydrogen diffusion flame. This relaxed residual stress caused by the welding and also cleaned some of the quartz vapor deposition on the fiber caused by the welding. Since losses at the fiber ends contribute disproportionately to the violin mode losses, this action probably has some influence.

Finally, we recognize that in building this suspension we have tried to incorporate as much as possible of the prior state of the art. Most suspensions should perform better than they do, given known loss mechanisms. It may simply be a matter of luck that the numerous unquantifiable errors in this line of work happened to be minimized in this particular experiment.

This research was undertaken as part of the development of advanced mirror suspensions for LIGO. While the quality factors measured represent an improvement in performance over earlier work, they do not yet demonstrate dissipation at the very low level required for an advanced LIGO, which is several times lower than reported here [30]. For this reason, as well as for the confirmation of NTE, continued research is necessary.

Acknowledgements

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Appendix A. Derivation of the dilution factor for a violin mode

We begin with the frequencies for the violin modes, Eq. (6). If we model the mechanical loss as an imaginary component to the Young’s modulus, $E = E_0(1 + i\phi)$, where $\phi \ll 1$, then the dilution factor $D_n$ appears in the imaginary part of the mode frequency [31]:

$$ f_n = f_{n0}(1 + i\phi/2D_n). \quad (A.1) $$
The Young’s modulus appears in Eq. (6) only through \( k_e = k_{00}/\sqrt{1+i\phi} \), so we see immediately that

\[
fn = \frac{n}{2L} \sqrt{\frac{P}{\rho L}} \left[ 1 + \frac{2}{k_{00}L} (1 + i\phi) \right] + \left( 4 + \frac{(n\pi)^2}{2} \right) \frac{1}{(k_{00}L)^2} (1 + i\phi),
\]

(A.2)

where, using the smallness of \( \phi \), we get

\[
fn \approx \frac{n}{2L} \sqrt{\frac{P}{\rho L}} \left[ 1 + \frac{2}{k_{00}L} (1 + i\phi/2) \right] + \left( 4 + \frac{(n\pi)^2}{2} \right) \frac{1}{(k_{00}L)^2} (1 + i\phi)
\]

(A.3)

\[
\approx \frac{n}{2L} \sqrt{\frac{P}{\rho L}} \left[ 1 + \frac{2}{k_eL} + \left( 4 + \frac{(n\pi)^2}{2} \right) \frac{1}{(k_eL)^2} \right] \times \left[ 1 + i\phi \left( 1 + \frac{1}{k_{00}L} + \left( 4 + \frac{(n\pi)^2}{2} \right) \frac{1}{(k_{00}L)^2} \right) \right]
\]

(A.4)

from which Eq. (8) follows by inspection.

### Appendix B. Derivation of the temperature dependence of the violin mode frequencies

We begin with Eq. (6). By inspection it is clear that the factor outside the braces is just the violin mode frequency for a perfect string with no resistance to bending. The factor after the 1 inside the braces is the correction to the frequencies due to the fiber stiffness. Re-expressing the frequency as

\[
fn = f_{00}[1 + g],
\]

(B.1)

we take the derivative of \( f_n \) with respect to temperature \( T \) and apply the chain rule to separate the two contributions:

\[
\frac{df_n}{dT} = \frac{df_{00}}{dT} + \frac{df_{\phi}}{dT}. \tag{B.2}
\]

The term \( f_{00} \) can depend on \( T \) through \( \rho_L \), since the linear mass density is inversely proportional to \( L \). Propagating these dependencies on \( T \) through gives

\[
\frac{df_{00}}{dT} = -\frac{1}{2}(\alpha - u_0\beta). \tag{B.5}
\]

We therefore see that if the strain \( u_0 \) of the fiber is such that the length of the fiber is invariant with temperature, then \( f_{00} \) is also invariant with temperature.

If we evaluate \( (dg/dT)/(1 + g) \), we get, since \( g \ll 1 \),

\[
\frac{dg}{dT} \approx \frac{dg}{dT}\frac{(1 + g)}{1 + g} = \frac{dL}{dT} \left[ 1 + \frac{2}{k_eL} + \left( 4 + \frac{(n\pi)^2}{2} \right) \frac{1}{k_eL^2} \right]. \tag{B.6}
\]

and by simple but tedious application of the chain rule we get

\[
\frac{dg}{dT} \approx \frac{2}{k_eL^2} \frac{dL}{dT} + \frac{2 + (n\pi)^2}{k_eL^3} \frac{dL}{dT} + L \frac{dk_e}{dT}, \tag{B.7}
\]

from which we factor out the dilution factor to get

\[
\frac{dg}{dT} \approx \frac{1}{D_n} \left[ \frac{dL}{dT} \frac{L}{k_e} + \frac{dk_e}{dT} \right]. \tag{B.8}
\]

In order to evaluate \( dg/dT \) further we must evaluate \( dk_e/dT \), where \( k_e \) is given by Eq. (4). We first note that, for this experiment, all the mode frequencies are so low that, to a very good approximation, \( k_e = \sqrt{P/E\ell} \). Therefore, \( k_e \) can vary with \( T \) only through \( E \) and \( I = \pi r^4/4 \). The fiber radius \( r \) is given by

\[
r = r_0 \left( 1 + \alpha T - \sigma(T) \frac{P}{E(T)A} \right), \tag{B.9}
\]

where \( \sigma \) is the Poisson’s ratio, which enters because the stretching of the fiber by the tension reduces its diameter. The Poisson’s ratio is also a temperature-dependent quantity. If we let \( \beta_\sigma = (d\sigma/dT)/\sigma \), then

\[
\frac{dr}{r} \approx (\alpha - u_0\beta + u_0\beta) \tag{B.10}
\]

Plugging this into our formula gives

\[
\frac{dg}{dT} \approx \frac{1}{D_n} \left[ -\alpha \beta + \beta + 2u_0\beta \right]. \tag{B.11}
\]

where \( u_0 = P/AE \). The term \( f_{00} \) can also depend on \( T \) through \( \rho_L \), since the linear mass density is inversely proportional to \( L \). Propagating these dependencies on \( T \) through gives

\[
\end{equation}

We therefore see that if the strain \( u_0 \) of the fiber is such that the length of the fiber is invariant with temperature, then \( f_{00} \) is also invariant with temperature.

If we evaluate \( (dg/dT)/(1 + g) \), we get, since \( g \ll 1 \),

\[
\frac{dg}{dT} \approx \frac{dg}{dT}\frac{(1 + g)}{1 + g} = \frac{dL}{dT} \left[ 1 + \frac{2}{k_eL} + \left( 4 + \frac{(n\pi)^2}{2} \right) \frac{1}{k_eL^2} \right]. \tag{B.6}
\]

and by simple but tedious application of the chain rule we get

\[
\frac{dg}{dT} \approx \frac{2}{k_eL^2} \frac{dL}{dT} + \frac{2 + (n\pi)^2}{k_eL^3} \frac{dL}{dT} + L \frac{dk_e}{dT}, \tag{B.7}
\]

from which we factor out the dilution factor to get

\[
\frac{dg}{dT} \approx \frac{1}{D_n} \left[ \frac{dL}{dT} \frac{L}{k_e} + \frac{dk_e}{dT} \right]. \tag{B.8}
\]

In order to evaluate \( dg/dT \) further we must evaluate \( dk_e/dT \), where \( k_e \) is given by Eq. (4). We first note that, for this experiment, all the mode frequencies are so low that, to a very good approximation, \( k_e = \sqrt{P/E\ell} \). Therefore, \( k_e \) can vary with \( T \) only through \( E \) and \( I = \pi r^4/4 \). The fiber radius \( r \) is given by

\[
r = r_0 \left( 1 + \alpha T - \sigma(T) \frac{P}{E(T)A} \right), \tag{B.9}
\]

where \( \sigma \) is the Poisson’s ratio, which enters because the stretching of the fiber by the tension reduces its diameter. The Poisson’s ratio is also a temperature-dependent quantity. If we let \( \beta_\sigma = (d\sigma/dT)/\sigma \), then

\[
\frac{dr}{r} \approx (\alpha - u_0\beta + u_0\beta) \tag{B.10}
\]

Plugging this into our formula gives

\[
\frac{dg}{dT} \approx \frac{1}{D_n} \left[ -\alpha \beta + \beta + 2u_0\beta \right]. \tag{B.11}
\]
What knowledge we have of $\beta_\sigma$ from the temperature dependencies of $E$ and of the shear modulus $\mu$ indicates that its value is comparable to $\beta$ and of the same sign [23]. Therefore, the terms involving $\sigma$ in this equation: (1) are small compared to those without $\sigma$, and (2) tend to cancel each other. We therefore drop them. The error in this approximation may be estimated by noting that, in this experiment, $\sigma \approx u_0 \beta$, and since $\sigma = 0.17$ for fused silica, the size of the terms involving $\sigma$ will be smaller than $\alpha$. Since $\alpha$ is 100 times smaller than $\beta/2$, the approximation is quite good, much better than the error in measuring the dilution factor.

The final answer is

$$\frac{df_n}{dT} \approx \frac{1}{2} (\alpha - u_0 \beta) + \frac{1}{D_n} [\alpha + u_0 \beta + \beta/2]. \quad (B.12)$$

References
[1] If Pythagoras himself wrote anything, none of it survives today. What we know now of his research in music comes from Nicomachus’s “The Manual of Harmonics”, written some 600 years later.
[17] This is not the only effect that can create vertical bounce sidebands, and might not be the only one present in our system. A description of other effects can be found in the thesis of Yinglei Huang, Syracuse University (1996).
[28] Our welding torch is made of copper, so we ourselves have not completely solved this problem.