Continuous measurements: partial selection

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Abstract

This Letter is devoted to the consideration of the continuous indirect measurement in the case of approximate measurement of the QRS observable. The equation of object matrix evolution is obtained. This equation is compared with the equation obtained by restricted path integral (RPI) method. The general solution of equation that has been obtained is found. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

For the past few decades, the theory of continuous measurements has received a great deal of attention. A number of various approaches to the description of continuous measurements have been put forward [2–7]. In this Letter, two of these approaches are examined. The first approach is based on restricted path integrals (RPI). It has been developed in detail by Mensky [2,3]. In the second approach the continuous measurement is proposed to be simulated as a sequence of single instantaneous measurements (see [6,8]).

In considering the problems related to measurements, it is usually assumed that the initial state of quantum read-out system (QRS) is pure and the measurement of QRS observable that is used to obtain estimation of object observable is precise. But in reality the QRS state might be mixed and the measurement of QRS observable is always approximate. The possibility of QRS to be in a mixed state was taken into account in [9]. The possibility of QRS observable measurement to be approximate was taken into account in [2, Chapter 3] (see also the case of orthogonal measurements in [10]). In both of these cases, the pure initial object state becomes mixed as a result of continuous monitoring of the object observable. According to the RPI method, the pure object state remains pure in the course of a continuous measurement process. So one can conclude that RPI method describes ideal continuous measurements that correspond to pure QRS state and precise measurement of QRS observable.

In the present Letter we are considering the case of the approximate QRS observable measurement with continuous monitoring of object measuring. Our goal is to obtain a differential equation that would describe the evolution of object under measurement and, ultimately, to solve this equation.
2. Equation that describes evolution of object under continuous indirect measurement

The peculiarity of the approach that is being used here consists in the fact that the continuous measurement is considered as a sequence of single instantaneous measurements separated by small time interval during which the object free evaluation takes place.

The state after single ideal measurement is determined by the relation [9] (see also the case of orthogonal measurements in [10])

$$\hat{\rho}(\bar{Y}) = \frac{\hat{R}(\bar{Y})\hat{\rho}_0\hat{R}^+(\bar{Y})}{\text{Tr}[\hat{R}^+(\bar{Y})\hat{R}(\bar{Y})\hat{\rho}_0]}$$ (1)

where $\bar{Y}$ is the result of precise measurement of QRS observable, $\hat{R}(\bar{Y})$ is the reduction operator, and $\hat{\rho}_0$ is the initial object state.

The state after measurement that is determined by (1) is normalized. However, it is more convenient to use unnormalized density matrices when dealing with questions concerned with continuous measurements [1]. The norm of such a matrix is equal to probability density of obtaining the measurement result or the sequence of measurement results. So, in order to obtain the expression for unnormalized density matrix, one has to multiply the right side of expression (1) by the probability density of the measurement result $\bar{Y}$ that is equal to Tr$[\hat{R}^+(\bar{Y})\hat{R}(\bar{Y})\hat{\rho}_0]$. Then one would have

$$\hat{\rho}(\bar{Y}) = \hat{R}(\bar{Y})\hat{\rho}_0\hat{R}^+(\bar{Y})$$ (2)

The matrices used in calculations below are the unnormalized density matrices.

Relation (2) can be generalized to take into account the case of approximate QRS observable measurement. When the measurement is approximate, the connection between the obtained value $\bar{Y}$ and actual value $Y$ is characterized by the conditional probability density $W(\bar{Y}|Y)$. The state after such a measurement is determined by relation

$$\hat{\rho}(\bar{Y}) = \int W(\bar{Y}|Y)\hat{R}(Y)\hat{\rho}_0\hat{R}^+(Y) dY.$$ (3)

When the measurement is precise, the conditional probability density is δ-function and relation (3) changes to relation (1).

Let us assume that a sequence of $N$ single measurements takes place: the measurement $j$ is at the time $t_j$ and between moments $t_{j-1}$ and $t_j$ the object state is being changed as a result of an action of the free evolution operator $\hat{U}_j$. Then the object state after measurement is determined by the following relation:

$$\hat{\rho}([\bar{Y}_j]) = \int \prod_{j=1}^{N} W(\bar{Y}_j|Y_j) \prod_{j=N}^{1} \hat{U}_j\hat{R}(Y_j)\hat{\rho}_0 \times \prod_{j=1}^{N} \hat{R}^+(Y_j)\hat{U}_j^+[dY_j].$$ (4)

where $[\bar{Y}_j]$ is the sequence of measurement results.

According to the standard measurement scheme, the reduction operator can be expressed in the following form:

$$\hat{R}(Y) = \Psi(Y - a_0\hat{A}),$$

where $\Psi(Y)$ is some function that satisfies condition $\int |\Psi(Y)|^2 dY = 1$, and $a_0$ is coefficient of interaction.

In order to obtain the continuous measurement out of a sequence of single measurements, the time intervals between consequent measurements of sequence must be decreased. In order to fix the inaccuracy of continuous measurement over a fixed time interval, value $a_0$ must be in inverse proportion to square root of measurement number over unit of time $\nu$:

$$a_0 = \frac{\alpha}{\sqrt{\nu}}.$$

Now, in order to obtain the differential equation for object density matrix, let us find a ratio of density matrix increment over the small time interval $\delta t$ to value of that interval. The increment of object density matrix has to be linearized over the interval $\delta t$.

During the process of linearization, the reduction operators $\hat{R}(Y_j)$ of expression (4) and the free evolution operators $\hat{U}_j$ appear as a power series with the small parameter $\tau/N$. Members of the zero order in expansion of the reduction operator, as well as of the free evolution operator, are numbers. Therefore, in a linear approximation, the multiplication of operators from expression (4) will become a sum. In case of a particle, such a result makes it possible to separate the terms that relate to the free evolution and the measurement process. So, the calculation of the change in object state can only be made with consideration of the influence of measurement process. The influence
of free evolution must be taken into account at the final stage.

If the interval of time $\delta t$ is very small, every measurement result from sequence $\{\tilde{Y}_j\}$ that was obtained during $\delta t$ has the same distribution. In particular, they have the same average, which depends on the value of the measured object observable $\hat{A}$, and they have the same finite dispersion. Dispersion of measurement result has a non-zero value, so the sequence of measurement results cannot be treated as values of some smooth function, because such a function would have infinite large fluctuations when $v$ tends to $\infty$. So in the recording part of device there must be a unit that would average measurement results over an interval of time $\tau$. To avoid additional influence of averaging process, the interval $\tau$ must be much shorter than the typical period of object evolution (see [2, Chapter 5] or [11] for details). For example, averaging interval $\tau$ might be equal to $\delta t$. In addition, the frequency of measurements has to be much greater than the value $1/\tau$. From mathematical point of view, it is equal to the double limit:

$$
\tilde{Y}(t) = \lim_{\tau \to 0} \lim_{v \to \infty} \frac{1}{\tau v} \sum_j \tilde{Y}_j \equiv \lim_{\tau \to 0} \tilde{Y}_\tau(t).
$$

Dispersion of random quantity $\tilde{Y}_\tau(t)$ tends to zero in limit $v \to \infty$ when $\tau$ is infinitely small but non-zero.

Let us assume that the averaging interval is equal to small interval of time $\delta t$. Function value $\tilde{Y}(t) \approx \tilde{Y}_\tau(t)$ can be realized by different sets of measurement results. Final density matrix $\hat{\rho}(\tilde{Y}(t), t + \delta t)$ is a mixture of density matrix $\hat{\rho}(\tilde{Y}_\tau, t + \delta t)$ (in calculations below the time is dropped to make relations shorter), each of these corresponds to the single sequence of measurements’ results that satisfies the condition

$$
\hat{\rho}(\tilde{Y}) = N \int_{\sum_j (\tilde{Y}_j - \tilde{Y}) = 0} \hat{\rho}([\tilde{Y}_j]) [d\tilde{Y}_j]. \tag{5}
$$

Since our goal is to watch the object observable $\hat{A}$, let us recalculate function $\tilde{Y}(t)$ to function $\tilde{\alpha}(t)$ that corresponds to an estimate of object observable $\hat{A}$:

$$
\tilde{\alpha}(t) = \frac{\sqrt{v}}{\alpha} \tilde{Y}(t).
$$

Density matrix $\hat{\rho}(\tilde{\alpha})$ differs from $\hat{\rho}(\tilde{Y})$ by normalization factor $\alpha/\sqrt{v}$. After rewriting relation (5), taking into account relation (4) and the rearrangement of multipliers, we will obtain

$$
\hat{\rho}(\tilde{\alpha}) = \frac{\alpha\sqrt{N}}{\sqrt{v}} \int \left( \sqrt{N} \int \prod_{j=1}^{N} W(\tilde{Y}_j|Y_j) [d\tilde{Y}_j] \right) \times \prod_{j=1}^{N} \tilde{R}(Y_j) \hat{\rho}_0 \prod_{j=1}^{N} \tilde{R}_+(Y_j) [dY_j]. \tag{6}
$$

If estimate of QRS observable is unbiased, then the average value of $\tilde{Y}_\tau$ is equal to $Y_\tau$. Let us also assume that conditional dispersion of $\tilde{Y}_\tau$ does not depend on $Y_j$, and is equal to $D$. The terms in the round brackets at the last expression represent nothing else but the probability density of random value $(1/\sqrt{N}) \times \sum_{j=1}^{N} (\tilde{Y}_j - Y_j)$. According to the limit theorem [12] the distribution of such a value tends to normal distribution with dispersion equal to $D$ and an average equal to zero when $N \to \infty$. As a result, we obtain the following:

$$
\hat{\rho}(\tilde{\alpha}) = \frac{\alpha\sqrt{N}}{\sqrt{v}} \sqrt{\frac{N}{2\pi D}} \int \exp \left[ -N \frac{(\tilde{Y} - Y_\Sigma)^2}{2D} \right] \times \prod_{j=1}^{N} \tilde{R}(Y_j) \hat{\rho}_0 \prod_{j=1}^{N} \tilde{R}_+(Y_j) [dY_j]. \tag{7}
$$

where $Y_\Sigma$ is equal to $(1/N) \sum_{j=1}^{N} Y_j$.

After a mathematical transformation (see Appendix A) one can have an expression for density matrix $\hat{\rho}([\tilde{\alpha}(t)], t + \delta t)$ at the limit $N \to \infty$:

$$
\hat{\rho}(\tilde{\alpha}) = \left[ \begin{array}{c}
0 \\
\frac{\alpha^2 / \sqrt{N}}{2\pi (\sigma Y + D)} \int \exp \left[ -\frac{\alpha^2 / \sqrt{N}}{4(\sigma Y + D)} \left( \frac{i}{\hbar} 2\sigma Y \left[ (\hat{A} - \tilde{\alpha})^2 \right] \right) \\
- \left[ (\hat{A} - \tilde{\alpha})^2 \right] \right]
\end{array} \right] \hat{\rho}_0. \tag{8}
$$
The norm of the density matrix $\hat{\rho}(\hat{a})$ is equal to the probability density of obtaining the single value $\hat{a}$ during the time interval $\delta t$. Our goal is for the norm to be equal to the probability of obtaining the curve $[\hat{a}(t)]$ during the time of monitoring $t$. It can be shown that the measure on the space of curves of this kind must be equal to

$$
\left(\frac{\alpha^2 \delta t}{2\pi(\sigma_Y + D)}\right)^{1/2} \frac{1}{\delta t^{1/2}} \prod_{j=1}^{n-1} d\hat{a}_j
$$

(see [13, Chapter 9] and [2, Appendix 3]). So, to write the term in right part of (10), which is proportional to $[\dot{A}, [\dot{A}, \hat{\rho}]]$. Because of this term, the state of object, that was initially pure, becomes mixed during the measurement process. One can see that this term appears because of inaccuracy of QRS observable measurement (conditional dispersion of estimation $D$ is non-zero) and because of averaging unit in recording part of the device as well. The only case when averaging unit does not lead to mixing of object state is when QRS state is in a squeezed base state, so that $\sigma_Y = \sigma_\rho = 1$.

The presence of correlation between measured QRS observable and an observable conjugated with it, leads to an additional action aimed at the object (real add-on to Hamiltonian $(\alpha^2/\sigma_\rho^2)(\sigma_\rho^2 - \sigma_Y^2)$). For example, if object is a mechanical oscillator and measured observable is a coordinate, the correlation leads to the appearance of an extra rigidity—negative or positive depending on the sign of the correlation $\dot{Y}$ and $\dot{P}_Y$. The questions concerned with the reverse action on an object as a result of device observables correlation are considered explicitly in [9] for the case of single measurements. Evidently, the influence of correlation can be eliminated by choosing the optimal combination of observable $\dot{Y}$ and $\dot{P}_Y$ as the QRS observable to be measured.

3. Solution of the equation

One can verify directly that relation

$$
\hat{\rho}([\hat{a}(t)], t) = \alpha \sqrt{\frac{D}{\hbar^2/4(\sigma_Y + D)}} \left(1 + \frac{\alpha^2 \delta t}{4(\sigma_Y + D)} \right)^{1/2} \frac{1}{\delta t} \prod_{j=1}^{n-1} d\hat{a}_j
$$

(9)

where $\hat{\rho}([\hat{a}(t)], t)$ is reduction operator of continuous indirect measurement, defined by relation

$$
\hat{R}(\xi, t) = \exp\left[-\frac{i}{\hbar} \int_0^t \left(\hat{H}_{\text{obj}}(\tau) - \frac{\alpha^2 \delta t}{2(\sigma_Y + D)} (\dot{\hat{a}}(\tau))^2 - \frac{\alpha^2 \delta t}{2(\sigma_Y + D)} \right) d\tau\right]
$$

(10)

The basic difference between Eq. (10) and equation with complex Hamiltonian discussed in [2,3] is the term in right part of (10), which is proportional to $[\dot{A}, [\dot{A}, \hat{\rho}]]$. Because of this term, the state of object, that was initially pure, becomes mixed during the measurement process. One can see that this term appears because of inaccuracy of QRS observable measurement (conditional dispersion of estimation $D$ is non-zero) and because of averaging unit in recording part of the device as well. The only case when averaging unit does not lead to mixing of object state is when QRS state is in a squeezed base state, so that $\sigma_Y = \sigma_\rho = 1$.

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$$

(10)
correlation in QRS. In order to demonstrate this, let us correspond to the intermediate case at the absence of
in the non-selective approach.

4. Comparison of the results with the ones obtained by restricted path integral (RPI) method

RPI method leads to the equations below, which describe the evolution of object whose observable is under continuous monitoring [2]:

\[
\frac{d}{dt} \hat{\rho} = -i \frac{\hbar}{\epsilon} \left[ \hat{H}_{\text{obj}}, \hat{\rho} \right] - k \left( \hat{A} - \hat{a}(t) \right)^2 \hat{\rho}
\]

in the selective approach and

\[
\frac{d}{dt} \hat{\rho} = -i \frac{\hbar}{\epsilon} \left[ \hat{H}_{\text{obj}}, \hat{\rho} \right] - k \left[ \hat{A}, \left[ \hat{A}, \hat{\rho} \right] \right]
\]

in the non-selective approach.

Differential equation (10) obtained in this Letter corresponds to the intermediate case at the absence of correlation in QRS. In order to demonstrate this, let us re-write it in the following form:

\[
\frac{d}{dt} \hat{\rho} = -i \frac{\hbar}{\epsilon} \left[ \hat{H}_{\text{obj}}, \hat{\rho} \right] - k (1 - \gamma) \left( \hat{A} - \hat{a}(t) \right)^2 \hat{\rho} - \frac{k}{2} \gamma \left[ \hat{A}, \left[ \hat{A}, \hat{\rho} \right] \right],
\]

where

\[
k = \frac{\alpha^2}{4\sigma_y}\frac{4\sigma_y\sigma_P}{\hbar^2}, \quad \gamma = \frac{D}{\sigma_y + D(4\sigma_y\sigma_P)}.
\]

Quantity \( \gamma \) varies from 0 to 1. When \( \gamma = 0 \), Eq. (15) turns to (13) that corresponds to selective approach, and when \( \gamma = 1 \), Eq. (15) turns to (14) that corresponds to a non-selective approach. Quantity \( \gamma \) has sense of measurement non-selectivity degree.

5. Conclusion

In this Letter we have obtained the equation describing the evolution of an object state, when its observable is under continuous indirect measurement for the case of the approximate measurement of QRS observable.

It has been demonstrated that non-selectivity which always causes initially pure object state to become mixed arises not only as a result of the fact that measurement of the QRS observable is approximate but as a result of the averaging of the measurement results in the laboratory device as well. The only case when the process of averaging does not cause non-selectivity is when QRS state is a squeezed base state.

If the measurement of QRS observable is precise, the QRS state is a squeezed base state and there is no correlation between measured QRS observable and observable conjugated with it, the equation obtained here coincides with the equation obtained by RPI method in the selective approach. If QRS observable measurement is approximate \((D \rightarrow \infty)\), the equation obtained here coincides with the equation obtained by RPI method in the non-selective approach independently of QRS state.

It has also been demonstrated that, when there is correlation between measured QRS observable and observable conjugated with it, an extra real add-on to object Hamiltonian arises.

The general solution for the equation has been found.

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Appendix A. The expression for density matrix at the limit $N \rightarrow \infty$

Function $\Psi(Y)$ can be expressed through its Fourier image $\psi(k)$ according to the relation [12]

$$\Psi(Y) = \frac{1}{\sqrt{2\pi}} \int \psi(k) \exp[-ikY] dk.$$ 

After substituting it in (7) one can obtain

$$\dot{\rho}(\tilde{\alpha}) = \frac{\alpha \sqrt{N}}{\sqrt{\nu}} \sqrt{\frac{N}{2\pi D}} \int \exp\left[-N\left(\frac{\tilde{\alpha}}{\sqrt{\nu}}\right)^2\right]$$

$$\times \frac{1}{(2\pi)^N} \left(\prod_{j=1}^{N} \psi(k_j^* \psi^*(k_j^*)\right)$$

$$\times \exp\left[-i \sum_{j=1}^{N} k_j^* \left(Y_j - \frac{\alpha}{\sqrt{\nu}} \hat{A}\right)\right] \hat{\rho}_0$$

$$\times \exp\left[i \sum_{j=1}^{N} k_j^* \left(Y_j - \frac{\alpha}{\sqrt{\nu}} \hat{A}\right)\right]$$

$$\times [dY_j][dk_j^*].$$ \hspace{1cm} (A.1)

Let us change the variables $[Y_j] \ (j = 1, N)$ to variables $[Y_j] \ (j = 1, N-1)$, $Y_N$. After integration by variables $[Y_j] \ (j = 1, N-1)$ and then by variables $[k_j^*] \ (j = 1, N-1)$, relation (A.1) is rearranged to

$$\dot{\rho}(\tilde{\alpha}) = \frac{\alpha \sqrt{N}}{2\pi} \sqrt{\frac{N}{2\pi D}} \int \exp\left[-N\left(\frac{\tilde{\alpha}}{\sqrt{\nu}}\right)^2\right]$$

$$\times \left(\prod_{j=1}^{N} \psi(k_j^* + k_N^* - k_N^0)\psi^*(k_j^0)\right)$$

$$\times \exp\left[i \frac{\alpha}{\sqrt{\nu}} \left(\frac{N}{2} k_N^* + k_N^* - k_N^0\right) \hat{A}\right] \hat{\rho}_0$$

$$\times \exp\left[-i \frac{\alpha}{\sqrt{\nu}} \left(\frac{N}{2} k_N^*\right) \hat{A}\right]$$

$$\times d\Psi dk_N^* [dk_N^*].$$ \hspace{1cm} (A.2)

Let us integrate by variable $Y_N$ and make a change of variable according to relations $k_N^* = k_N + \kappa / 2\sqrt{N}$. As a result, (A.2) is rearranged to

$$\dot{\rho}(\tilde{\alpha}) = \frac{\alpha \sqrt{\nu}}{2\pi} \int \exp\left[-\frac{\kappa^2 D}{2} - i\alpha \sqrt{\nu} \tilde{\alpha}\right]$$

$$\times \left(\prod_{j=1}^{N} \psi\left(k_j + \frac{\kappa}{2\sqrt{N}}\right)\psi^*\left(k_j - \frac{\kappa}{2\sqrt{N}}\right)\right)$$

$$\times \exp\left[i \frac{\alpha}{\sqrt{\nu}} \left(\frac{N}{2} k_j \hat{A}\right) + i \frac{\alpha}{\sqrt{\nu}} \hat{A}\right]$$

$$\times dk [dk_j].$$

(A.3)

where $\delta \Omega(\hat{\rho})$ is linear operator that acts in linear space of finite Hermitian matrix. It is defined by relation

$$\delta \Omega(\hat{\rho}) = \int \psi(k_j + \frac{\kappa}{2\sqrt{N}})\psi^*(k_j - \frac{\kappa}{2\sqrt{N}})$$

$$\times \exp\left[i \frac{\alpha}{\sqrt{\nu}} k_j \hat{A}\right] \hat{\rho} \exp\left[-i \frac{\alpha}{\sqrt{\nu}} k_j \hat{A}\right].$$

(A.4)

Now let us change $\nu$ to $N/\delta t$, make an expansion terms under integral in $1/\sqrt{N}$ power series with Peano remainder up to second order of vanishing. After integrating the relation obtained, taking into account relations

$$\int |\psi(k)|^2 dk = 1,$$

$$\frac{1}{2} \int (\psi^* \psi^{*}(k) - \psi^* \psi(k)) dk = i \tilde{Y},$$

$$\int |\psi(k)|^2 k dk = \frac{i}{\hbar} \tilde{F}_Y,$$

$$\int |\psi'(k)|^2 k dk = \frac{1}{2} \int (\psi^* \psi^{*}(k) - \psi^* \psi(k)) dk = \sigma_Y + \tilde{Y}^2,$$

$$\frac{i}{2} \int (\psi^* \psi^{*}(k) - \psi^* \psi(k)) k dk = \frac{\sigma_{FP} + \tilde{F}_Y \tilde{Y}}{\hbar},$$
\[ \int |\psi(k)|^2 k^2 dk = \frac{\sigma_\rho + \frac{\sigma_Y^2}{\hbar^2}}{\hbar^3}. \quad (A.5) \]

one can have expansion of \( \delta \Omega(\hat{\rho}) \) in \( 1/\sqrt{N} \) power series with Peano remainder up to second order of vanishing:

\[
\delta \Omega(\hat{\rho}) = \left( 1 + i \left( \kappa \hat{Y} + \alpha \sqrt{\delta t} \hat{P}_Y \right) \frac{1}{\sqrt{N}} \right)
- \left( \frac{\kappa^2}{2} \sigma_Y - \kappa \alpha \sqrt{\delta t} \frac{\sigma_Y \sigma_P}{\hbar} \{ \hat{A}, . \} \right)
+ \frac{\alpha^2 \delta t \sigma_P}{2 \hbar^2} \{ \hat{A}, \{ \hat{A}, . \} \} + o \left( \frac{1}{N} \right) \hat{\rho}.
\]

(A.6)

The only case when there is limit of \( \delta \Omega^N(\cdot) \) is when the vanishing term of first order in expansion (A.6) (that is proportional to \( 1/\sqrt{N} \)) is equal to zero. It is possible when \( \hat{Y} = 0 \) and \( \hat{P}_Y = 0 \). So, relation (A.3) at the limit \( N \to \infty \) can be rearranged to

\[
\hat{\rho}(\tilde{a}) = \sqrt{\frac{\alpha^2 \delta t}{2\pi (\sigma_Y + D)}}
\times \exp \left[ \frac{\alpha^2 \delta t}{4(\sigma_Y + D)} \left( \frac{i}{\hbar} 2\sigma_Y \left( \hat{A} - \tilde{a} \right)^2 \right) \right]
- \left\{ \left( \hat{A} - \tilde{a} \right)^2 \right\}
\]

\[ - \frac{1}{2} \left( \frac{D}{\sigma_P h^2/4} \frac{\sigma_Y \sigma_P - \sigma_Y^2 \sigma_P - h^2/4}{h^2/4} \right)
\times \{ \hat{A}[\hat{A}, .] \} \hat{\rho}_0. \quad (A.7) \]

References